Finite Supermodular Design with Interdependent Valuations*

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Abstract

This paper studies supermodular mechanism design in environments with arbitrary (finite) type spaces and interdependent valuations. In these environments, the designer may have to use Bayesian equilibrium as a solution concept, because ex-post implementation may not be possible. We propose direct (Bayesian) mechanisms that are robust to certain forms of bounded rationality while controlling for equilibrium multiplicity. In quasi-linear environments with informational and allocative externalities, we show that any Bayesian mechanism that implements a social choice function can be converted into a supermodular mechanism that also implements the original decision rule. The proposed supermodular mechanism can be chosen in a way that minimizes the size of the equilibrium set, and we provide two sets of sufficient conditions to this effect. This is followed by conditions for supermodular implementation in unique equilibrium.

Keywords: Implementation, mechanisms, multiple equilibrium problem, learning, strategic complementarities, supermodular games.

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1 Introduction

This paper studies supermodular mechanism design in environments with interdependent valuations and arbitrary (in particular, multidimensional) finite type spaces. This approach was introduced by Mathevet (26) in differentiable quasilinear environments with private values and one-dimensional types.[1] The main motivation is to design direct mechanisms that are robust to certain forms of bounded rationality while controlling for equilibrium multiplicity. It is important to extend supermodular mechanism design to environments with informational and allocative externalities and multidimensional types for at least two reasons. First, these environments capture many realistic situations. Second, it is often impossible to use dominant strategy or ex-post implementation in these settings (see Jehiel *et al.* (22) and Section 2), and thus the designer may resort to Bayesian equilibrium as a solution concept. It becomes useful to have a simple method for improving the behavioral robustness of Bayesian mechanisms.

In this paper, we are concerned with the design of supermodular mechanisms whose equilibrium set is of minimal size. We call this minimal supermodular implementation. Supermodular mechanism design aims to induce the right incentives so that agents play a supermodular game. Supermodular games are games where players have monotone best responses, i.e. each player wants to play a "larger" strategy if others do so as well. On the theoretical front, the reasons for using supermodular mechanisms stem from Milgrom and Roberts (28), (29) and Vives (34): supermodular games have extremal equilibria, a smallest and a largest one, that enclose all the iteratively undominated strategies and all the limit points of all adaptive and sophisticated learning dynamics. Therefore, supermodular games are robust to a wide range of behaviors, including boundedly rational behaviors. In particular, if the designer had the opportunity to use her mechanism repeatedly, then adaptive learners (Milgrom and Roberts (28)) would end up within the interval prediction, which is the interval between the extremal equilibria. Therefore, the objective of minimizing the size of the interval prediction has several virtues. It minimizes the multiple equilibrium problem, since all equilibria are contained in it.[2] It also guarantees a more accurate convergence of the learning dynamics. Ideally, this interval reduces to a single point in certain situations (see Section 2), thereby solving the multiplicity issue and ensuring convergence of all dynamics.

Supermodular mechanisms have other attractive theoretical properties. Not only are their mixed strategy equilibria unstable (Echenique and Edlin (17)), which justifies

[1] Chen (8) was the first to propose a supermodular mechanism (to implement the Lindahl correspondence).

[2] If the outcome function of the mechanism is continuous and if the interval prediction is tight, then all equilibrium outcomes are close, so that the output of the mechanism must be close to the socially desirable objective.

ruling them out of the analysis, but many pure equilibria are stable, such as the extremal equilibria (Echenique (16)). Thus, a perturbation should not destabilize permanently a socially desirable alternative implemented via a supermodular mechanism (provided the underlying equilibrium was stable).

The robustness properties of supermodular mechanisms have been corroborated by several experiments. Chen and Gazzale (11) run experiments on a game for which they control the amount of supermodularity. They show how convergence in that game is significantly better when it is supermodular. Healy (18) tests five public goods mechanisms and he observes that subjects learn to play the equilibrium only in those mechanisms that induce a supermodular game. Other experiments (e.g. Chen and Plott (9), and Chen and Tang (10)) provide results emphasizing the importance of dynamic convergence in the context of implementation. Most of these experiments demonstrate that convergence to an equilibrium is not a trivial issue.

In this paper, we generalize supermodular mechanism design to environments with allocative externalities, interdependent valuations (i.e. informational externalities) and arbitrary (finite) type spaces. There are two important reasons for doing so.

Firstly, it allows us to cover mechanism design problems of interest. The importance of allocative externalities is well documented in the literature. Jehiel and Moldovanu (20) use patent licensing in an oligopolistic market as an example. Informational externalities are also a realistic assumption, proved to be interestingly challenging by many papers (Cremer and McLean (13), Maskin (25), Dasgupta and Maskin (14), Perry and Reny (31), Chung and Ely (12), Bergemann and Morris (2), etc). Finally, it is often natural to interpret information as a multidimensional type in many design problems. Consider, for example, oil companies bidding to obtain a drilling permit. Their private information is modeled as a multidimensional signal (e.g., expected amount of oil in the oil field, proximity to other reserves, etc).

Secondly, it is difficult to use dominant strategy or ex-post implementation in those environments — with allocative externalities, interdependent valuations and multidimensional types — and thus behaviorally-robust Bayesian mechanisms become especially appealing. In quasilinear environments with interdependent valuations and multidimensional types, many impossibility results limit the set of available solution concepts. The conclusions are rather pessimistic about dominant strategy equilibrium and ex-post equilibrium. Williams and Radner (35) show that efficient dominant strategy implementation is generally not possible when agents have interdependent valuations. Jehiel *et al.* (22) prove a strong impossibility result: when types are multidimensional and valuations are interdependent, only trivial decision rules can generically be implemented in ex-post equilibrium. If the designer wants to implement a meaningful social

choice function, not even necessarily efficient, she may have to use Bayesian equilibrium as a solution concept (see Section 2). Even then, impossibility results exist. Jehiel and Moldovanu (21) show that it is difficult to reconcile Bayesian incentive compatibility with some efficiency constraint. These negative results indicate that Bayesian equilibrium may often be a natural candidate as a solution concept. However, playing a Bayesian equilibrium requires more, in general, on the part of the agents. Agents have to be Bayesian rational, and strong knowledge assumptions about the information structure and the rationality must hold (Brandenburger and Dekel (5)). For example, Bayesian equilibrium calls for correct predictions of opponents' play to determine one's own strategy. In this context, the ability to construct supermodular Bayesian mechanisms is attractive, because eventual play of some equilibrium can be achieved by unsophisticated agents who follow simple behavioral rules.

Our paper provides methods for converting *any* truthful Bayesian mechanism into a (truthful) supermodular mechanism whose equilibrium set is of minimal size. The idea is to create complementarities between agents' announcements by augmenting the original transfer scheme with a function. This function vanishes in expectation and therefore preserves incentive compatibility. Although there exist many ways in which a mechanism can be transformed into a supermodular mechanism, we derive the one that most adequately addresses the multiple equilibrium problem. To this purpose, we add just enough strategic complementarities to ensure that a supermodular game is induced, but not in any excess of that level.

We present two sets of results for minimal supermodular implementation. In both instances, "best" is used to designate the mechanism with the smallest interval prediction. The first result shows that if a social choice function is implementable, then its decision rule can be implemented by the best supermodular mechanism among all the supermodular mechanisms *whose transfers are in a certain class*. No additional condition is required. In particular, this result holds for all (implementable) decision rules and all valuation functions. The result also provides an explicit transfer scheme.

The second result characterizes the *overall* best supermodular mechanism among all possible supermodular mechanisms or transfers: if a social choice function is implementable, then its decision rule can be implemented by the (overall) best supermodular mechanism if and only if some (explicitly stated) finite system of linear equations admits a solution. This result determines the existence and the numerical values of the minimally supermodular transfers. As a complementary result, we provide a sufficient condition, order reducibility, under which existence of a solution is guaranteed. Although the former result reaches a weaker conclusion than the latter, it applies under very general conditions.

Finally, we provide conditions that ensure that truthtelling is the (essentially) unique equilibrium. For fine type spaces, this guarantees stability: all learning dynamics converge to the truthful equilibrium and the game is dominance solvable.

Beyond the generalizations of supermodular mechanism design, this paper provides new insights into the design of minimally supermodular mechanisms. The use of finite types clarifies the existence and the construction of these mechanisms. For example, the fact that minimally supermodular mechanisms are a solution to a system of linear equations allows the application of numerical methods for designing them. Further, this formulation helps clarify what the necessary conditions for the existence of such transfers might look like. In this regard, we provide a simple sufficient condition under which the system admits a solution; although this condition is not necessary, it gives valuable information about the type of mechanism design problems that might cause issues for minimally supermodular implementation.

A number of papers are related to our work. The first paper to present a supermodular mechanism was Chen (8). Mathevet (26) developed supermodular mechanism design as a general method under incomplete information. Since his paper is the closest to ours, our contribution deserves clarification. As already said, our environment is more general, due to the interdependent values and the multidimensional types, but we use a finite setup. In Mathevet (26)'s environment, dominant strategy implementation is still a powerful tool, while it in our environment it is significantly less applicable. The present paper also clarifies the construction of minimally supermodular mechanisms, especially with our reducibility condition and the formulation as a linear system. Finally, we propose different options for minimal supermodular design when sufficient conditions fail, while Mathevet (26) does not. In particular, our first main result always applies. Cabrales and Serrano (7) study implementation with boundedly rational agents who follow adaptive better response dynamics. This learning rule excludes learning processes that our mechanisms allow for. Finally, our paper is related to the literature on rationalizable implementation (e.g. Abreu and Matsushima (1), and Bergemann, Morris, and Tercieux (4)), because this solution concept has the potential to imply nice learning properties when a unique equilibrium is rationalizable. Abreu and Matsushima (1) show that any social choice function can be virtually implemented in iteratively undominated strategies. Their result is very powerful but their mechanism remains complex, as the dimension of the message space must be arbitrarily large. Experimental evidence does not support this mechanism (Sefton and Yavas (32)). Instead we look at direct mechanisms and exact implementation. In general, the concept of rationalizability is such that a strategy may not be rationalizable, because it is dominated by another dominated strategy, an argument a la Jackson (19). For ex-

ample, in Bergemann, Morris, and Tercieux (4), the best responses are not well-defined off-equilibrium, and off-equilibrium behaviors are one of our motivations.

The remainder of the paper is organized as follows. A motivating example is presented in Section 2. Section 3 defines the framework of supermodular mechanism design. Section 4 introduces the notion of minimal implementation and contains our two main results. Section 5 studies supermodular implementation in unique equilibrium. Concluding remarks appear in Section 6. All proofs are relegated to the Appendix.

2 Motivating Example

This section illustrates our approach in a simple example. In this example, the designer would like to make an efficient decision, but this cannot be done in ex-post equilibrium and, hence, also not in dominant strategies. Thus, the designer may decide to work with Bayesian implementation.[3] We show how the designer can start from *any* truthful Bayesian mechanism, in particular one with poor stability properties, and turn it into a (truthful) supermodular mechanism with a unique equilibrium.

Consider a social planner who has to make a decision between two public projects, A: improving the roadway or B: building a park. There are two agents (e.g., food vendors), 1 and 2, on each side of the road who will be affected by the decision. Each agent i receives a two-dimensional signal $\theta_i = (\theta_i^A, \theta_i^B) \in \{(1,2),(2,1)\}$, where each dimension represents a projected flow of traffic resulting from the implementation of the respective project. Types are drawn with equal probabilities: $\Pr(\theta_1 = (1,2)) = \Pr(\theta_2 = (1,2)) = .5$. Let $V_i^g(\theta)$ be i's valuation for project $g \in \{A, B\}$ at $\theta = (\theta_1, \theta_2)$. Valuations represent the expected gain for each vendor from a project and are given in the following matrix (rows represent agent 1):

$V(\theta)$	$(1,2)$	$(2,1)$
$(1,2)$	$V_1^A = 1 \quad V_2^A = 1$ $V_1^B = 2.5 \quad V_2^B = -1$	$V_1^A = 2 \quad V_2^A = 0$ $V_1^B = 1.5 \quad V_2^B = 1$
$(2,1)$	$V_1^A = 2 \quad V_2^A = 0$ $V_1^B = 1.5 \quad V_2^B = 1$	$V_1^A = 4 \quad V_2^A = -2$ $V_1^B = 1 \quad V_2^B = -1.$

Agent 1 always interprets the higher flow of traffic as good news in terms of revenue, and that view is reinforced by agent 2's signal. That is, 1's value for project A (B)

[3]There are many situations in which Bayesian implementation is possible, while ex-post implementation is not. So, we are not restricting attention to non-generic cases. Bayesian incentive compatibility (BIC) can be viewed as a weighted average of ex-post incentive constraints. Thus, some ex-post constraints can be violated while BIC holds, if other ex-post constraints hold with enough slack.

increases in the first (second) dimension of her signal and of agent 2's signal. From 1's perspective, there are positive informational externalities coming from agent 2.

Agent 2, on the other hand, considers the first dimension of both signals to be bad news for project A. In particular, he thinks that the higher the increase in traffic due to improved roadway, the lower its profits will be, because increased traffic flow should attract other competitors. That is, 2's value for A decreases in the first dimension of his signal and of 1's signal. As for project B, 2 thinks that it will be profitable for him only if there are medium levels of traffic, which corresponds to opposite estimates from 1 and 2, $\theta_i^B \neq \theta_j^B$. When both second dimensions are low, the park should not increase the market size and the profits; the same applies if both estimates are high, due to (expected) increased competition. Overall, from 2's perspective, the informational externalities coming from 1 are ambiguous (they are negative for project A and ambiguous for B).

The designer's objective is to choose the efficient project, i.e., the project which maximizes the sum of valuations for each possible profile of types. Given the valuations, the efficient decision rule, which the designer would like to implement, is:

$x(\hat{\theta})$	$(1,2)$	$(2,1)$
$(1,2)$	A	B
$(2,1)$	B	A

Denote agent i's transfers as a function of reported types by:

$t_i(\hat{\theta})$	$(1,2)$	$(2,1)$
$(1,2)$	t_i^1	t_i^2
$(2,1)$	t_i^3	t_i^4

The efficient decision rule is not ex-post incentive compatible.[4] To see why, let us consider the ex-post incentive compatibility conditions for agent 1 when agent 2's type is $\theta_2 = (1,2)$ and is truthfully reported. At type $\theta_1 = (1,2)$, ex-post incentive compatibility for agent 1 requires

$$1 + t_1^1 \geq 2.5 + t_1^3.$$

At type $\theta_1 = (2,1)$, ex-post incentive compatibility requires

$$1.5 + t_1^3 \geq 2 + t_1^1.$$

The last two inequalities cannot be jointly satisfied, which proves that the efficient decision rule is not ex-post implementable. The designer is therefore inclined to work

[4]Ex-post incentive compatibility requires that for all i and θ, $u_i(x(\theta),\theta) \geq u_i(x(\theta_i',\theta_{-i}),\theta)$ for all θ_i'. This means that if all other agents report truthfully, truthtelling is a best response for each agent i at every possible realizations of types θ.

with Bayesian implementation. We proceed to show that there exist transfers that implement the efficient decision rule in Bayesian equilibrium.

Bayesian incentive compatibility for agent 1 requires that truthtelling be weakly preferred to lying when her true type is (1, 2)

$$.5(1 + t_1^1) + .5(1.5 + t_1^2) \geq .5(2.5 + t_1^3) + .5(2 + t_1^4)$$

and when her true type is (2, 1)

$$.5(1.5 + t_1^3) + .5(4 + t_1^4) \geq .5(2 + t_1^1) + .5(1 + t_1^2).$$

Combining these two inequalities, we obtain that for any t_1 such that

$$2.5 \geq t_1^1 + t_1^2 - t_1^3 - t_1^4 \geq 2$$

the efficient decision rule is Bayesian incentive compatible for agent 1. Similarly, Bayesian incentive compatibility is satisfied for any t_2 such that

$$0 \geq t_2^1 - t_2^2 + t_2^3 - t_2^4 \geq -3.$$

In particular, the designer can choose:

$t_1(\hat{\theta})$	(1, 2)	(2, 1)
(1, 2)	-5	7
(2, 1)	4.75	-5

$t_2(\hat{\theta})$	(1, 2)	(2, 1)
(1, 2)	5	-5
(2, 1)	-5.5	5

As we will see, the magnitude of these transfers is large enough to offset any consideration about the valuations. Given these transfers, the resulting payoff matrix in the ex ante Bayesian game is:

EU	truthtelling	constant (1,2)	constant (2,1)	always lie
truthtelling	2.43*; .1*	1.25; 0	3.5; -.5	2.31; -.62
constant (1,2)	2.37; .25	-2.75; 4.75*	8.62*; -5	3.5*; -.5
constant (2,1)	2.37;-.75	6.37*; -5.5	-2.75; 4.75*	1.25; 0
always lie	2.31; -.62	2.37; -0.75	2.37; 0.25*	2.44; .12

where row entries and first payoffs correspond to agent 1, while column entries and second payoffs correspond to agent 2. Best responses are denoted by asterisks. The game described by this payoff matrix is not (ex ante) dominance solvable. Despite being the unique equilibrium, truthtelling is unstable; after a small perturbation, convergence to

it fails under various dynamics. The intuition goes as follows. If agent 2 plays the constant announcement $(1, 2)$ irrespective of his true type, then agent 1 will best-respond by announcing (2, 1) regardless of her type. In return, agent 2 will also announce (2, 1) for every type. Then agent 1 will want to play the constant announcement (1, 2), followed by a constant announcement of (1, 2) by agent 2. We are back to the original strategy of agent 2. These transfers give rise to cycling behaviors and the problem extends beyond best-response dynamics.

To overcome this problem, we propose converting the mechanism into a supermodular mechanism. The idea is to modify the original transfers in a way that adds complementarity between agents' reports. Start from the above transfers t_i. Given a collection of numbers $\{\delta_i(\theta) : \theta \in \Theta\}$, define

$$t_i^{SM}(\hat{\theta}) = \delta_i(\hat{\theta}) - E_{\theta_{-i}}[\delta_i(\hat{\theta}_i, \theta_{-i})] + E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})] \tag{2.1}$$

for each agent i. Transfers t_i^{SM} satisfy $E_{\theta_{-i}}[t_i^{SM}(\theta_i, \theta_{-i})] = E_{\theta_{-i}}[t_i(\theta_i, \theta_{-i})]$ for all θ_i, i.e., these two transfer functions have the same expected value when agents other than i report their type truthfully. Thus, if agent i finds it optimal to play truthfully under t_i (when others do so), then she also finds it optimal under t_i^{SM}. Thus, transfers t^{SM} elicit truthful revelation for every collection $\{\delta_i(\theta) : \theta \in \Theta\}$. Choosing this collection is the next question. In differentiable environments, supermodularity is characterized by positive cross-partial derivatives (Milgrom and Roberts (28)). If our environment were differentiable, we would have $\partial^2 t_i^{SM}/\partial\hat{\theta}_i\partial\hat{\theta}_j = \partial^2\delta_i/\partial\hat{\theta}_i\partial\hat{\theta}_j$. That is, δ_i controls the supermodularity, which is also true in our environment. In Section 4, we provide the formula for an appropriate δ_i. Since an agent's utility is $V_i + t_i^{SM}$, the formula essentially adds enough supermodularity to compensate any effects coming from V_i, but not in any excess. When applied to this example, the formula and (2.1) output

$t_1^{SM}(\hat{\theta})$	$(1,2)$	$(2,1)$
$(1,2)$	1	1
$(2,1)$	-1.625	1.375

$t_2^{SM}(\hat{\theta})$	$(1,2)$	$(2,1)$
$(1,2)$	-.25	-1
$(2,1)$	-.25	1

which translates into the ex ante payoff matrix:

EU	truthtelling	constant (1,2)	constant (2,1)	always lie
truthtelling	2.44*; .13*	1.06; 0	3.69*; -.5	2.31; -.63
constant (1,2)	2.38; −.38*	3.25*;-.5	2.63; -1	3.5*; -1.13
constant (2,1)	2.38; -.13	0; -.25	3.63; .75*	1.25; .63
always lie	2.31; -.63	2.19; -.75	2.56; .25*	2.44; .13

This payoff matrix describes a supermodular game — assuming $(2,1) > (1,2)$ — in

which truthtelling is the unique equilibrium. Supermodularity implies that, for every true type, each agent wants to make larger announcements (under the assumed order) if the other agent does so as well. This mechanism has nice properties: iterative deletion of strictly dominated strategies gives a unique prediction, truthtelling, and all adaptive learning dynamics converge to the truthful equilibrium (wherever they are initiated) by Milgrom and Roberts (28). The original instability problem is solved.

In Section 7.1 of the Appendix, we present another version of this example where the designer starts with transfers that create multiple equilibria. In this case, our transformation technique solves the multiple equilibrium problem, as truthtelling is the unique equilibrium of our supermodular mechanism.

3 Finite Supermodular Design: The Framework

Consider n agents, each endowed with quasilinear preferences over a set of alternatives. The set of agents will be denoted by N. An alternative is a vector $(x,t) = (x_1,\ldots,x_n,t_1,\ldots,t_n)$, where $x_i \in X_i$ and $t_i \in \mathbb{R}$ for all i. In this environment, x_i is interpreted as agent i's allocation; $x \in X \equiv \prod_{i=1}^{n} X_i$ is the complete allocation profile; t_i is the money transfer to i; and $t \in \mathbb{R}^n$ is the vector of transfers.

Each agent i has a finite type space Θ_i with generic element θ_i. The types of agents other than i are denoted by $\theta_{-i} \in \Theta_{-i} \equiv \prod_{j\neq i} \Theta_j$, and $\theta \in \Theta \equiv \prod_{i\in N} \Theta_i$ denotes a full type profile. There are no restrictions on the nature of the type spaces: each Θ_i could be, for example, a subset of $\mathbb{R}$, $\mathbb{R}^n$, or any other finite collection of elements. Information is incomplete. There is a common prior with probability mass function ϕ on Θ known to the mechanism designer. Types are assumed to be independently distributed, and ϕ has full support.

A mechanism designer wishes to implement an allocation for each realization of types. This objective is represented by a decision rule $x : \Theta \mapsto (x_i(\theta))_{i=1}^{n}$. To this end, the designer sets up a transfer scheme $t_i : \Theta \to \mathbb{R}$ for each i. A mechanism is denoted by $\Gamma = (\{\Theta_i\},(x,t))$. Agents are asked to announce a type, and from the vector of announced types, an allocation and a transfer accrue to each agent. The pair $f = (x,t)$ is called a social choice function. We adopt the conventional notation where $\hat{\theta}_i$ is agent i's announced type, $\hat{\theta}_{-i}$ is the vector of announced types of all agents but i, and $\hat{\theta}$ denotes the announced types of all agents.

Each agent i's preferences over alternatives are represented by a utility function $u_i(x,t_i,\theta) = V_i(x;\theta) + t_i$, where $V_i : X \times \Theta \to \mathbb{R}$ is referred to as i's valuation. This formulation allows for allocational externalities, as V_i can depend on the allocations of agents other than i. It also captures the case of informational externalities (inter-

dependent valuations) since the valuations may depend on other agents' types. Agent i's utility function at type θ in Γ is $u_i^\Gamma(\hat{\theta};\theta) = V_i(x(\hat{\theta});\theta) + t_i(\hat{\theta})$. A pure strategy for agent i under incomplete information is a function $\hat{\theta}_i : \Theta_i \to \Theta_i$ that maps true types into announced types. Strategy $\hat{\theta}_i(\cdot)$ is called a deception. Agent i's (ex ante) utility function in Γ is $U_i^\Gamma(\hat{\theta}_i(.), \hat{\theta}_{-i}(.)) = E_\theta[u_i^\Gamma(\hat{\theta}(\theta);\theta)]$.

A partial order $\geq$ on a set X is a binary relation that satisfies reflexivity, antisymmetry, and transitivity (see Topkis (33)). The couple $(X, \geq)$ is referred to as a partially ordered set. For $x, y \in X$, if $y \geq x$ and $y \neq x$, then we write $y > x$. A total order on set X is a binary relation that satisfies comparability, antisymmetry, and transitivity.[5] If $\geq$ is a total order on X, then $(X, \geq)$ is called a totally ordered set. An interval in $(X, \geq)$ is a set of the form $[x', x''] = \{x \in X : x'' \geq x \geq x'\}$.

A total order $\geq^*$ on set X is called a linear extension of the partial order $\geq$ if (i) $(X, \geq^*)$ is a totally ordered set and (ii) for every x, y in X, if $y \geq x$, then $y \geq^* x$. Elements that are ordered under $\geq$ remain identically ordered under $\geq^*$, but $\geq^*$ also orders all the elements that are unordered under $\geq$. By the order-extension principle (Marczewski (24)), every partial order can be extended to a total order. Therefore, every partially ordered set admits a linear extension.

Suppose that $(X, \geq_X)$ and $(Y, \geq_Y)$ are partially ordered sets. A function $h : X \times Y \to \mathbb{R}$ has increasing (decreasing) differences in (x, y) on $X \times Y$ if for all $x'' \geq_X x'$ and all $y'' \geq_Y y'$, $h(x'', y'') - h(x', y'') \geq (\leq) h(x'', y') - h(x', y')$. Increasing (decreasing) differences express the notion of strategic complementarities (substitutes) when applied to payoff functions.

Take x, x' in a partially ordered set $(X, \geq)$. If x and x' have a least upper bound (greatest lower bound) in X, it is referred to as their join (meet) and denoted by $x \vee x'$ $(x \wedge x')$. A lattice is a partially ordered set that contains the join and meet of every pair of its elements. Given a lattice X, a function $h : X \to \mathbb{R}$ is supermodular if $h(x) + h(x') \leq h(x \vee x') + h(x \wedge x')$ for all x and x' in X.

A finite game is a tuple $(N, \{(\Sigma_i, \geq_i)\}, \{w_i\})$ where N is a finite set of players; $(\Sigma_i, \geq_i)$ is a finite partially ordered strategy set for each i; and $w_i : \prod_{i \in N} \Sigma_i \to \mathbb{R}$ is Player i's payoff function. In the following definition, the set of strategy profiles of i's opponents, denoted $\Sigma_{-i} = \prod_{j \neq i} \Sigma_j$, is endowed with the product order induced by $\{\geq_j\}_{j \neq i}$, according to which σ'_{-i} is weakly larger than σ_{-i} iff $\sigma'_j \geq_j \sigma_j$ for all $j \neq i$.

DEFINITION **1** *A finite game $\mathcal{G} = (N, \{(\Sigma_i, \geq_i)\}, \{w_i\})$ is supermodular if for all $i \in N$, (1) $(\Sigma_i, \geq_i)$ is a lattice, (2) w_i has increasing differences in (σ_i, σ_{-i}) on (Σ_i, Σ_{-i}), and (3) w_i is supermodular in σ_i on Σ_i for each $\sigma_{-i} \in \Sigma_{-i}$.*

[5] Comparability means that $x \geq y$ or $y \geq x$ for all x, y in X. Note that comparability implies reflexivity; hence, every total order is also a partial order.

The paper focuses on totally ordered strategy sets. In this case, requirements (1) and (3) are automatically satisfied and we only need to satisfy (2) to ensure that the game is supermodular.

We endow agents' type spaces with ordering relations and use them to define supermodular implementation. For all i, let $(\geq_i^1, \geq_i^2)$ be a pair of orders such that $\geq_i^1$ is a total order on Θ_i and $\geq_i^2$ is a total order on Θ_{-i}. Denote by $\geq_{-i}$ the product order induced by $\{\geq_j^1\}_{j\neq i}$ on Θ_{-i}. A profile of orders $\{(\geq_i^1, \geq_i^2)\}$ is said to be *consistent* if for all i, $\geq_i^2$ is a linear extension of $\geq_{-i}$ on Θ_{-i}.

The game induced by mechanism Γ can be formulated at three stages: Ex ante, interim, and ex-post (complete information). Among these three formulations, the paper considers supermodularity at the ex-post level, because this is the strongest requirement. If the ex-post game is supermodular for all possible realizations of types θ, then the game will be supermodular in its ex ante and interim formulations.

Let $\mathcal{G}(\theta) = (N, \{(\Theta_i, \geq_i^1)\}, \{u_i^\Gamma(\cdot\,;\theta)\})$ denote the game induced ex-post by mechanism Γ. Let $\mathcal{G} = (N, \{(\Theta_i^{\Theta_i}, \geq_i)\}, \{U_i^\Gamma\})$ be the ex ante Bayesian game induced by Γ, where $\geq_i$ is the pointwise order induced by $\geq_i^1$ on $\Theta_i^{\Theta_i}$. The next definition introduces the main implementation concept in the context of direct mechanisms.

Definition **2** *A social choice function $f = (x, t)$ is (truthfully) supermodular implementable if truthtelling, i.e., $\hat{\theta}_i(\theta_i) = \theta_i$ for all i, is a Bayesian equilibrium of $\mathcal{G}$ and if $\mathcal{G}(\theta)$ is supermodular for each θ.*

4 Minimal Supermodular Implementation

In this section, we present two results dealing with minimally supermodular mechanisms. The main issue with supermodular implementation lies in finding the appropriate amount of complementarity to add to a mechanism. While complementarities lead to good dominance and learning properties, via the monotonicity of the best responses, excessive complementarities can generate multiple equilibria. Therefore, one negative consequence might be enhancing the "learnability" of undesirable equilibria. In our model, only the truthful equilibrium outcome is known to be socially desirable. Hence, adding complementarities can make it easier for agents to learn; but if this induces multiplicity, they may learn to play an untruthful equilibrium.

This section is organized as follows. First, we present the foundational results and concepts behind minimal supermodular implementation. Then we present our results in separate sections. Our first result is that, for *any* implementable social choice function, its decision rule can be minimally supermodular implemented by transfers within a class. This result holds for *all* valuation functions. This is a strong result, conditionally on

choosing transfers within the class. Our second result does not restrict attention to a class of mechanisms or transfers. For *any* valuation functions, if a social choice function is implementable, then its decision rule can be minimally supermodular implemented (among *all* transfers that induce a supermodular game) if and only if a finite system of linear equations admits a solution. This solution, if it exists, is the set of minimally supermodular transfers. As a complementary result, we provide a sufficient condition, called order reducibility, under which existence of a solution is guaranteed. Both results provide explicit expressions for the transfers.

4.1 Foundations

Mathevet (26) relates the degree of complementarities to the size of the equilibrium set via the following binary relation.

DEFINITION **3** *The binary relation $\succeq_{ID}$ on the space of transfer functions is defined such that $\tilde{t} \succeq_{ID} t$ if for all $i \in N$ and for all $\theta_i'' \geq_i^1 \theta_i'$ and $\theta_{-i}'' \geq_{-i} \theta_{-i}'$, $\tilde{t}_i(\theta_i'', \theta_{-i}'') - \tilde{t}_i(\theta_i'', \theta_{-i}') - \tilde{t}_i(\theta_i', \theta_{-i}'') + \tilde{t}_i(\theta_i', \theta_{-i}') \geq t_i(\theta_i'', \theta_{-i}'') - t_i(\theta_i'', \theta_{-i}') - t_i(\theta_i', \theta_{-i}'') + t_i(\theta_i', \theta_{-i}')$.*

This binary relation orders transfers according to the magnitude of their increasing differences. In differentiable environments, this definition is equivalent to saying that $\tilde{t} \succeq_{\text{ID}} t$, if and only if, for all $i \in N$ the cross-partial derivatives of $\tilde{t}_i$ are larger than those of t_i, $\partial^2 \tilde{t}_i(\theta)/\partial\theta_i\partial\theta_j \geq \partial^2 t_i(\theta)/\partial\theta_i\partial\theta_j$, for all $j \in N$ and $\theta \in \Theta$. This definition captures the amount of complementarities contained in transfers and compares them accordingly. While relation $\succeq_{\text{ID}}$ is transitive and reflexive, it is not antisymmetric. Denote the set of $\succeq_{\text{ID}}$-equivalence classes of transfers by $\mathcal{T}$.[6]

In a supermodular game, the interval prediction is the interval between the largest and the smallest equilibrium. We compare supermodular mechanisms by the size of the interval prediction of the game that they induce. The next proposition, taken from Mathevet (26), provides the tool to do so. If transfers t'' generate more complementarities than t', and if both transfers induce truthtelling and a supermodular game, then the interval prediction induced by t'' contains that induced by t'.

For any $t \in \mathcal{T}$ such that $f = (x, t)$ is supermodular implementable, let $\overline{\theta}_t^i : \Theta_i \to \Theta_i$ and $\underline{\theta}_t^i : \Theta_i \to \Theta_i$ denote the largest and the smallest equilibrium strategy of Player i in the game induced by the mechanism. Define

$$[\underline{\theta}_t, \overline{\theta}_t] = \{(s_1, \ldots, s_n) | s_i : \Theta_i \to \Theta_i \text{ and } \overline{\theta}_t^i(\theta_i) \geq_i^1 s_i(\theta_i) \geq_i^1 \underline{\theta}_t^i(\theta_i) \text{ for all } i \in N \text{ and } \theta_i \in \Theta_i\} \quad (4.1)$$

[6]Each equivalence class contains transfer functions t and $\tilde{t}$ such that $\tilde{t} \succeq_{\text{ID}} t$ and $t \succeq_{\text{ID}} \tilde{t}$ while $t \neq \tilde{t}$. Any quasi-order can be transformed into a partial order by using equivalence classes.

to be the interval of strategy profiles in between the extremal equilibria.

PROPOSITION 1 *If (x, t'') and (x, t') are supermodular implementable social choice functions and if $t'' \succeq_{ID} t'$, then $[\underline{\theta}_{t'}, \overline{\theta}_{t'}] \subset [\underline{\theta}_{t''}, \overline{\theta}_{t''}]$.*

This proposition is proved in Mathevet (26). It implies that the objective of minimizing the equilibrium set coincides with the objective of minimizing the complementarities. A social choice function $f = (x, t^*)$ will be minimally supermodular implementable if the transfers t^* elicit truthful revelation and induce a supermodular game with the weakest complementarities. This will give the tightest interval prediction around the truthful equilibrium.

4.2 Minimal Implementation under Total Orders

This section addresses minimal supermodular implementation within a class of transfers. We explicitly show how to convert any truthful mechanism into a supermodular mechanism while controlling for the intensity of the complementarities.

Our approach takes advantage of the totality of orders $\geq_i^1$. If the strategy sets are totally ordered, then the only requirement to check to satisfy Definition 1 is the increasing differences condition. Therefore, if the transfer functions ensure that (I) for each θ and i, $u_i^\Gamma(\hat{\theta}, \theta)$ has increasing differences in $(\hat{\theta}_i, \hat{\theta}_{-i})$ on $(\Theta_i, \geq_i^1) \times (\Theta_{-i}, \geq_{-i})$, then (II) the ex-post game $\mathcal{G}(\theta)$ will be supermodular for each θ, as desired. In this section, we restrict attention to the class of transfers that guarantee that (III) for each θ and i, $u_i^\Gamma(\hat{\theta}, \theta)$ has increasing differences in $(\hat{\theta}_i, \hat{\theta}_{-i})$ on $(\Theta_i, \geq_i^1) \times (\Theta_{-i}, \geq_i^2)$, where $\{(\geq_i^1, \geq_i^2)\}_i$ is a consistent profile of orders. Since $\geq_i^2$ is a linear extension of $\geq_{-i}$, (III) implies (I), and hence (II) holds. Consider the following family of transfers:

DEFINITION 4 *Family $\mathcal{F}(x, \{(\geq_i^1, \geq_i^2)\}_i) \subset \mathcal{T}$ is the set of transfers t such that (x, t) is truthfully implementable and $u_i^\Gamma(\hat{\theta}, \theta)$ has increasing differences on $(\Theta_i, \geq_i^1) \times (\Theta_{-i}, \geq_i^2)$ for each $\theta \in \Theta$ and $i \in N$, where $\{(\geq_i^1, \geq_i^2)\}_i$ is consistent.*

We now define our concept of minimal supermodular implementation.

DEFINITION 5 *A social choice function $f = (x, t^*)$ is minimally supermodular implementable over family $\mathcal{F}$ if it is supermodular implementable, $t^* \in \mathcal{F}$, and $t \succeq_{ID} t^*$ for all transfers $t \in \mathcal{F}$.*

Minimally supermodular transfers elicit truthful revelation and produce the supermodular game with the weakest complementarities. By Proposition 1, they give the tightest equilibrium set within the class of transfers $\mathcal{F}$. Here is our first main result.

THEOREM **1** *If $f = (x, t)$ is implementable, then for any consistent profile of orders $\{(\succeq_i^1, \succeq_i^2)\}_i$ there exist t^* such that (x, t^*) is minimally supermodular implementable over $\mathcal{F}(x, \{(\succeq_i^1, \succeq_i^2)\}_i)$.*

The theorem reaches a strong conclusion: for *any* implementable social choice function, its decision rule can be minimally supermodular implemented. There are no other restrictions on the decision rule or the valuation functions. Despite the finiteness of the type sets, there are infinitely many transfers that can supermodularly implement a decision rule for a given consistent profile of orders. Having a method for choosing the best among them is useful. To understand this, as well as our construction, start from any truth-revealing transfers $\{t_i(\theta) : \theta \in \Theta\}$. For each i and collection of numbers $\{\delta_i(\theta) : \theta \in \Theta\}$, define

$$t_i^*(\theta_i, \theta_{-i}) = \delta_i(\theta_i, \theta_{-i}) - E_{\theta_{-i}}[\delta_i(\theta_i, \theta_{-i})] + E_{\theta_{-i}}[t_i(\theta_i, \theta_{-i})]. \tag{4.2}$$

Transfers $t_i^*(\theta)$ satisfy $E_{\theta_{-i}}[t_i^*(\theta_i, \theta_{-i})] = E_{\theta_{-i}}[t_i(\theta_i, \theta_{-i})]$ for all θ_i, i.e., these two transfer functions have the same expected value when agents other than i report their type truthfully. Since agent i finds it optimal to play truthfully under $t_i(\theta)$ (when others do so), she must also find it optimal under $t_i^*(\theta)$. We conclude that for every collection $\{\delta_i(\theta) : \theta \in \Theta\}$, the transfers $t_i^*(\theta)$ also elicit truthful revelation. The problem becomes the choice of each $\delta_i(\theta)$, as there are infinitely many ways of inducing a supermodular game given a profile of orders. The proof provides an explicit formula for the collection $\{\delta_i(\theta) : \theta \in \Theta\}$ that generates the best transfers $\{t_i^*\}$ (within a family of transfers) in terms of minimizing the interval prediction.

To sum up, our method suggests totally ordering type sets and then using our formula. Can this method be useful? In Section 2, it delivered a supermodular mechanism with a unique equilibrium, while ex-post implementation was not an option. In the Appendix (see Section 7.1), it also delivers a supermodular mechanism with a unique equilibrium, while the original transfers produce multiple equilibria.

Given a choice of consistent orders, the theorem provides appropriate transfers. But there are many possible orders and the designer may want to discriminate among the many associated transfers. Suppose that the designer has a concept of distance, i.e., a metric d on Θ. Then Theorem 1 can be used to select the transfers that lead to the smallest equilibrium set across *all* the families. Let $\mathcal{F}^*(x)$ be the union of $\mathcal{F}(x, \{(\succeq_i^1, \succeq_i^2)\}_i)$ over all consistent orders $\{(\succeq_i^1, \succeq_i^2)\}_i$.

COROLLARY **1** *If $f = (x, t)$ is implementable, then there exist transfers t^{**} and consistent orders $\{(\succeq_i^{*1}, \succeq_i^{*2})\}_i$ such that (x, t^{**}) is minimally supermodular implementable over $\mathcal{F}(x, \{(\succeq_i^{*1}, \succeq_i^{*2})\}_i)$ and t^{**} give the smallest interval prediction in $\mathcal{F}^*(x)$ given d.*

Our corollary ultimately says that for every metric, there is a choice of total orders $(\geq_i^{*1}, \geq_i^{*2})$ for each i that is most adapted to d, since the equilibrium set resulting from the corresponding minimal transfers is minimized (under d) among all of $\mathcal{F}^*(x)$. The explanation is simple. For each profile of orders, the theorem provides the transfers that deliver the smallest interval prediction within the corresponding class. Since there are finitely many types, there are finitely many (consistent) profiles of orders. Therefore, there must be a profile of orders whose associated transfers deliver the smallest interval prediction under d among all of $\mathcal{F}^*(x)$.

4.3 Unconditional Minimal Implementation

In this section, we study (unconditional) minimal supermodular implementation by looking for the overall best transfers. In the previous section, the supermodular transfers were minimal within a class. We required that, for every agent i, increasing differences be satisfied on $(\Theta_i, \geq_i^1) \times (\Theta_{-i}, \geq_i^2)$. By doing so, we did not consider all the transfers that induce a supermodular game. Indeed, some transfers may induce increasing differences on $(\Theta_i, \geq_i^1) \times (\Theta_{-i}, \geq_{-i})$ but not on the above product set, yet it is sufficient for our purpose. This happens because $\geq_{-i}$ typically orders fewer elements than $\geq_i^2$, which changes the number of inequalities that have to hold to satisfy increasing differences. To summarize, our previous theorem was a conditional form of minimal supermodular implementation, while in this section, we aim for an unconditional form. For convenience, we write $V_i(x, \theta) = V_i(x_i, \theta)$ for all i to emphasize the dimension of the decision rule on which i's utility depends. This notation does not exclude allocative externalities, for an agent's own allocation x_i could be a function of another agent's allocation.

Definition **6** *A social choice function $f = (x, t^*)$ is minimally supermodular implementable if it is minimally supermodular implementable over family $\mathcal{T}$.*

Our main result shows that the problem of finding minimally supermodular transfers is equivalent to solving a system of linear equations. This insight is highly useful, as it allows the application of standard methods and algorithms from numerical linear algebra (e.g. Paige and Saunders (30), Demmel (15)). In what follows, we refer to the *supermodularity* of a function $h_i : \Theta \to \mathbb{R}$ between types $\theta_i'' \geq_i^1 \theta_i'$ and $\theta_{-i}'' \geq_{-i} \theta_{-i}'$ as the expression

$$h_i(\theta_i'', \theta_{-i}'') - h_i(\theta_i'', \theta_{-i}') - h_i(\theta_i', \theta_{-i}'') + h_i(\theta_i', \theta_{-i}'). \tag{4.3}$$

Consider (4.2) and note that the supermodularity of t_i^* is equal to the supermodularity of δ_i. Therefore, our objective of finding the overall best transfers is tantamount

to finding a function δ_i that induces increasing differences on $(\Theta_i, \geq_i^1) \times (\Theta_{-i}, \geq_{-i})$ for each i, without introducing unnecessary complementarities. Before deriving the linear system that corresponds to this objective, we define the concept of immediate successor/predecessor.

DEFINITION **7** *For x' and x'' in a partially ordered set $(X, \geq_X)$, x'' is an immediate successor of x' (and x' is an immediate predecessor of x'') if (a) $x'' >_X x'$, and (b) the set $\{x \in X | x'' >_X x >_X x'\}$ is empty.*

Consider the system of linear equations $A_i \cdot \delta_i = b_i$, where δ_i is a column vector that contains the values of $\delta_i(\theta)$ for every $\theta \in \Theta$; A_i is a sparse matrix whose nonzero elements (four per row) are equal to -1 or 1, and positioned so as to produce the supermodularity of δ_i for types that are immediate successors; b_i is a vector containing expressions (4.4), i.e., the minima of the differences in valuations between immediate successors. For example, a typical row of A_i takes the form $(\mathbf{0}, 1, \mathbf{0}, -1, \mathbf{0}, -1, \mathbf{0}, 1, \mathbf{0})$ where $\mathbf{0}$ is a block of zeroes, so that the dot product with vector δ_i produces an expression such as (4.3). The system matches this expression with an entry of b_i that involves the same types. In Section 7.2, we derive this system in a particular example.

The next proposition characterizes the (unconditional) minimally supermodular transfers as a solution to the above system of linear equations.

PROPOSITION **2** *Minimally supermodular transfers exist, if and only if, the finite linear system $A_i \cdot \delta_i = b_i$ has a solution $\delta_i \geq 0$ for all player i.*

Before providing the intuition for the result, we make a few remarks.

Assuming $\delta_i \geq 0$ is without loss of generality, because we can always add any positive constant c to any solution δ_i and obtain another solution. The reason is that any constant gets canceled out when we form the supermodularity of a function. Moreover, assuming $b_i \geq 0$ is also without loss of generality, since any equation with a negative right hand side can be multipled by -1 (i.e. the relevant rows of A_i and b_i get multiplied by -1). We can use linear programming (LP) techniques and software to solve the feasibility problem implied by the system of equations stated in Proposition 2. By introducing a vector z_i of positive auxiliary variables, we can restate the feasibility problem into a standard form LP:

$$\begin{aligned} &\min_{\delta_i, z_i} e^T \cdot z_i \\ \text{s.t. } &A_i \cdot \delta_i + z_i = b_i \\ &\delta_i \geq 0, z_i \geq 0 \end{aligned}$$

where e is a vector of 1s of appropriate dimension. This auxilary LP has optimal value 0 (i.e. $z = 0$) if and only if there exists a non-negative solution δ_i to the system of linear equations stated in the proposition.

Besides linearity, the main virtue of Proposition 2 is to only involve immediate successors. The necessity part of this result is intuitive. For all θ_i', θ_i'' where θ_i'' is an immediate successor of θ_i' in Θ_i, and for all θ_{-i}', θ_{-i}'', where θ_{-i}'' is an immediate successor of θ_{-i}' in Θ_{-i}, the supermodularity of δ_i (which corresponds to that of t_i^*) must equal

$$-\min_{\theta\in\Theta}[V_i(x_i(\theta_i'',\theta_{-i}''),\theta)-V_i(x_i(\theta_i'',\theta_{-i}'),\theta)-V_i(x_i(\theta_i',\theta_{-i}''),\theta)+V_i(x_i(\theta_i',\theta_{-i}'),\theta)], \quad (4.4)$$

for every i, for otherwise we could construct alternative transfers $\tilde{t}_i$ that satisfy this equality (at particular types); therefore, it could not hold that $\tilde{t} \succeq_{\text{ID}} t^*$, and hence no t^* could be minimally supermodular.

While necessity seems clear, it is not obvious that it suffices to search for δ_i's whose supermodularity equals (4.4) for successive types only. This property, which greatly simplifies the problem, comes from the proof of Theorem 1: the supermodularity of any function of two variables, when measured between non-successive elements, is equal to the sum of the supermodularities between all pairs of immediate successors in between. The intuition goes as follows. Take a function h with two variables, where each variable is in $\mathbb{N}$. Note that

$$h(2,3)-h(2,1)-(h(1,3)-h(1,1)) \quad (4.5)$$

is equal to

$$[h(2,3)-h(2,2)-(h(1,3)-h(1,2))]+[h(2,2)-h(2,1)-(h(1,2)-h(1,1))]. \quad (4.6)$$

The differences between non-successive types (1 and 3 are not immediate successors in (4.5)) are sums of differences between successive types, (4.6). Therefore, if the supermodularity of δ_i between successive types equals (4.4), which is the minimal requirement, then our previous observation implies that the supermodularity of δ_i between non-successive types must also be minimal. In conclusion, we just need to be concerned with supermodularity between successive types.

In Section 7.4, we provide a sufficient condition that ensures minimal transfers exist and have a simple closed-form representation. Although this condition is not necessary, it appears to be tight. The condition, called *order reducibility*, is imposed on the set of decision rules.

5 Uniqueness

In this section, we provide sufficient conditions for supermodular implementation in unique equilibrium. In light of our current results, a natural question to ask is: when does a minimally supermodular mechanism, i.e., one with the smallest equilibrium set, actually have a unique equilibrium? If a supermodular game has a unique equilibrium, then it is dominance-solvable, and many learning dynamics converge to the unique equilibrium (Milgrom and Roberts (28)). Supermodular implementation is, therefore, particularly appealing when truthtelling is the unique equilibrium. The study of unique supermodular implementation allows us to draw some conclusions regarding the type of environments — preferences and social choice functions — for which supermodular implementation may be most useful.

In this section, we impose more structure on the type sets. Suppose (Θ_i, d_i) is a metric space for every i. Our only requirement is that for any i, if $\theta_i' \geq_i^1 \theta_i$ and if $\theta_i'' \geq_i^1 \theta_i'$, then $d_i(\theta_i'', \theta_i) \geq d_i(\theta_i', \theta_i)$, so that each metric respects the order.

Recall that i's utility at type θ is denoted by $u_i^\Gamma(\hat{\theta}; \theta) = V_i(x(\hat{\theta}); \theta) + t_i(\hat{\theta})$. For each i and θ, let $K_i(\theta)$ be a real number such that

$$(u_i^\Gamma(\theta_i'', \theta_{-i}''; \theta) - u_i^\Gamma(\theta_i', \theta_{-i}''; \theta)) - (u_i^\Gamma(\theta_i'', \theta_{-i}'; \theta) - u_i^\Gamma(\theta_i', \theta_{-i}'; \theta)) \\ \leq d_i(\theta_i'', \theta_i') K_i(\theta) \sum_{j \neq i} d_j(\theta_j'', \theta_j') \quad (5.1)$$

for all $\theta_i'' \geq_i^1 \theta_i'$ and $\theta_{-i}'' \geq_{-i} \theta_{-i}'$.[7] Due to the finiteness of types, K_i always exists. When types are real-valued and d_i is the Euclidean metric, (5.1) can be written as

$$\frac{u_i^\Gamma(\theta_i'', \theta_{-i}''; \theta) - u_i^\Gamma(\theta_i', \theta_{-i}''; \theta)) - (u_i^\Gamma(\theta_i'', \theta_{-i}'; \theta) - u_i^\Gamma(\theta_i', \theta_{-i}'; \theta))}{(\theta_i'' - \theta_i') \sum_{j \neq i} (\theta_j'' - \theta_j)} \leq K_i(\theta), \quad (5.2)$$

and so if the environment is differentiable, $K_i(\theta) = \max_{j \neq i} \max_{\hat{\theta}} \partial^2 u_i^\Gamma(\hat{\theta}; \theta) / \partial \hat{\theta}_i \partial \hat{\theta}_j$.[8] The cross-partial derivatives measure the strategic complementarities between agents' reports. Therefore, $K_i(\theta)$ is an upper bound on the strategic complementarities (between agents' reports) induced by Γ at θ, and thus it is a nonnegative number. Note that K_i is an *endogenous* quantity, as it depends on the transfers.

[7] In (5.1), we use the L_1-norm induced by $\{d_i\}$ to measure the distance between opponents' profiles.

[8] Let $\theta_k'' = \theta_k'$ for $k \neq i, j$. If $\theta_i'' \to \theta_i'$ and $\theta_j'' \to \theta_j'$, the lhs of (5.2) becomes $\partial^2 u_i^\Gamma(\theta'; \theta) / \partial \hat{\theta}_i \partial \hat{\theta}_j$.

Similarly, for each i and θ_{-i}, let $\gamma_i(\theta_{-i})$ be a real number such that

$$(V_i(x(\hat{\theta}_i'', \theta_{-i}); \theta_i'', \theta_{-i}) - V_i(x(\hat{\theta}_i', \theta_{-i}); \theta_i'', \theta_{-i})) - (V_i(x(\hat{\theta}_i'', \theta_{-i}); \theta_i', \theta_{-i}) - V_i(x(\hat{\theta}_i', \theta_{-i}); \theta_i', \theta_{-i})) \geq \gamma_i(\theta_{-i}) d_i(\hat{\theta}_i'', \hat{\theta}_i') d_i(\theta_i'', \theta_i') \quad (5.3)$$

for all $\hat{\theta}_i'' \geq_i^1 \hat{\theta}_i'$ and $\theta_i'' \geq_i^1 \theta_i'$. Due to the finiteness of types, γ_i always exists. In differentiable environments with real-valued types, $\gamma_i(\theta_{-i}) = \min_{(\theta_i, \hat{\theta}_i)} \partial^2 V_i(x(\hat{\theta}_i, \theta_{-i}); \theta) / \partial \hat{\theta}_i \partial \theta_i$. Therefore, γ_i is a lower bound on the complementarities between i's own report and type when other agents report truthfully. Note that γ_i is an *exogenous* quantity, because it is determined by the primitives of the problem. Examples 3 and 4 in Mathevet (26) provide numerical illustrations for the computation of the K_i's and γ_i's.

Numbers K_i and γ_i represent opposite forces in the shaping of the equilibrium set. An agent with a large γ_i is very responsive to her own type, because her marginal utility is very sensitive to a change in θ_i. Thus, small changes in type lead to large changes in report, independently of others' reports, which have relatively little importance. Therefore, when γ_i is large, i's behavior is not responsive to others. By definition, equilibrium multiplicity is caused by the mutual influence that players have on one another. Since a large γ_i isolates i from the other agents, this favors uniqueness. A large K_i, however, expresses strong interdependence between players' reports. If i's behavior is strongly responsive to others, this favors multiplicity. These effects and their impact on the set of rationalizable strategy profiles in incomplete information games are the subject of Mathevet (27).

Our next results formalize the trade-off between these forces. Denote the truthful strategy by $\theta_i^T(\cdot)$. For each i and θ_i, let $\bar{K}_i(\theta_i) = \max_{\theta_{-i}} K_i(\theta)$.

PROPOSITION **3** *Let f be a supermodular implementable social choice function. For every i, consider some deception $\theta_i^*(\cdot) \geq \theta_i^T(\cdot)$. If there exist i, θ_i and $\hat{\theta}_i \in [\theta_i, \theta_i^*(\theta_i))$ such that*

$$\bar{K}_i(\theta_i) \sum_{j \neq i} E_{\theta_j}[d_j(\theta_j^*(\theta_j), \theta_j] - E_{\theta_{-i}}[\gamma_i(\theta_{-i})] d_i(\hat{\theta}_i, \theta_i) < 0 \quad (5.4)$$

then $\theta^*(\cdot)$ is not a Bayesian equilibrium. The same conclusion applies to deceptions $\theta_i^*(\cdot) \leq \theta_i^T(\cdot)$ if there exist i, θ_i and $\hat{\theta}_i \in (\theta_i^*(\theta_i), \theta_i]$ such that (5.4) holds.

The proof demonstrates that if agent i deviates from his report at type θ_i, $\theta_i^*(\theta_i)$, and announces $\hat{\theta}_i$ instead, while other agents play according to $\theta_{-i}^*(\cdot)$, then the lhs of (5.4) is an upper bound on how much i *loses* from that deviation. If this upper bound is positive, then the deviation may not be profitable. However, if it is negative, then $\hat{\theta}_i$

is a profitable deviation and hence $\theta^*(\cdot)$ cannot be an equilibrium.[9]

Proposition 3 should read as follows: if there exist metrics d_i and constants $K_i(\theta)$ and $\gamma_i(\theta_{-i})$ satisfying (5.1) and (5.3), and if the hypotheses of Proposition 3 are satisfied, then $\theta^*(\cdot)$ is not a Bayesian equilibrium. Of course, agents need not be aware of which metrics are used by the designer. In the same way, agents need not be aware of which order is used by the designer to induce a supermodular game.

Inequality (5.4) summarizes the trade-off between the opposite forces $\bar{K}_i$ and $E[\gamma_i(\cdot)]$. If the uniqueness effect dominates, i.e., $E[\gamma_i(\cdot)]$ is large enough, then the untruthful profile $\theta^*(\cdot)$ does not fall within the bounds of the interval prediction.[10]

The proposition is not useful for profiles for which, for every i and θ_i, $\theta_i^*(\theta_i)$ and θ_i are equal or successive types. In that case, $d_i(\hat{\theta}_i, \theta_i)$ would be zero. Furthermore, although the theorem is useful to determine whether a given strategy profile is not an equilibrium, it does not deliver an immediate conclusion about uniqueness. The next proposition addresses this question.

Before proceeding, we define a measure of coarseness on agents' type spaces. For any type θ_i in Θ_i, letting θ_i' and θ_i'' be its immediate predecessor and immediate successor, we define

$$\varepsilon_i(\Theta_i) = \max_{\theta_i \in \Theta_i} \max\{d_i(\theta_i', \theta_i), d_i(\theta_i, \theta_i'')\}$$

to be a measure of the maximal distance between any type in Θ_i and its immediate successor or predecessor. As we get closer to the continuous case, $\varepsilon_i(\Theta_i) \to 0$. Define $\varepsilon(\Theta) = \max_i \varepsilon_i(\Theta_i)$ to be the overall measure of coarseness.

Our next result will be concerned with "eventual uniqueness," which first requires to define the neighborhood of truthtelling.

DEFINITION 8 *A profile $\theta^*(\cdot)$ is outside the neighborhood of truthtelling if $\theta^*(\cdot)$ and $\theta^T(\cdot)$ are ordered and if for all i, there is θ_i such that $(\theta_i \wedge \theta_i^*(\theta_i), \theta_i \vee \theta_i^*(\theta_i)) \neq \emptyset$.*

A profile is outside the neighborhood of truthtelling if it is larger or smaller than truthtelling (i.e., all agents always over- or under-report), and if all agents i have the option for some θ_i to report a non-truthful type in between truth θ_i and her actual report $\theta_i^*(\theta_i)$. In order for an agent to have this option, her original deception must be far enough from truthtelling, for otherwise the only possible deviation would be to report her type truthfully.

[9] It is clear what losing 2 means, for example. But losing -2 is equivalent to gaining 2. Thus, when (5.4) holds, the loss must be negative and so the deviation profitable.

[10] Using the smallest possible $\bar{K}_i(\theta_i)$ and the largest possible $\gamma_i(\theta_{-i})$ is a natural way for the designer to utilize this proposition.

PROPOSITION 4 *Let f be a supermodular implementable social choice function (on Θ). If for every agent i*

$$(n-1)E_{\theta_i}[\bar{K}_i(\theta_i)] < E_{\theta_{-i}}[\gamma_i(\theta_{-i})], \tag{5.5}$$

then any profile $\theta^*(\cdot)$ outside the neighborhood of truthtelling such that for some θ

$$\min_i \left\{ \sum_{j \neq i} E_{\theta_j}[d_j(\theta_j, \theta_j^*(\theta_j))] \right\} > \varepsilon(\Theta)\xi(\theta) \tag{5.6}$$

where

$$\xi(\theta) = \frac{\max_i \{E_{\theta_{-i}}[\gamma_i(\theta_{-i})]\}}{\min_i \left\{ \frac{E_{\theta_{-i}}[\gamma_i(\theta_{-i})]}{n-1} - E_{\theta_i}[K_i(\theta_i)] \right\}}, \tag{5.7}$$

is not a Bayesian equilibrium.

This proposition says that if (5.5) holds, then the size of the equilibrium set depends essentially on the richness of the type sets. Indeed, if function ξ in (5.7) is bounded above as $\varepsilon(\Theta) \to 0$,[11] then $\varepsilon(\Theta)\xi(\theta) \to 0$. Thus, the set of profiles that include truthtelling and that might be equilibria vanishes as type sets become infinitely fine, and hence truthtelling eventually becomes unique. To be clear, condition (5.5) captures the main driving force behind uniqueness, but we must take the fineness of type sets into account. Otherwise, some untruthful profiles can become equilibria simply because some deviations are not available to an agent who would have otherwise chosen it. Note also that if (5.6) holds for some profile $\theta^*(\cdot)$, then it also holds for all finer type sets and all profiles $\theta^{**}(\cdot) \geq \theta^*(\cdot)$, assuming ξ is stable. Therefore, as $\varepsilon(\Theta) \to 0$, the set of potential equilibria shrinks monotonically to zero measure.

This proposition generalizes Mathevet (26)'s uniqueness result (Proposition 3, p.418) to our environments. In continuous type spaces, richness is obviously not an issue — (5.6) holds automatically when $\varepsilon(\Theta) = 0$ — and only (5.5) matters.

This proposition seems to be mostly useful a posteriori. After the mechanism has been built, we can use it to check whether truthtelling is eventually unique. However, it would be useful to know *a priori* whether the design problem at hand is compatible with unique supermodular implementation given its primitives.

Since the minimally supermodular transfers minimize the size of the equilibrium set, they are a natural choice for unique implementation. We have a closed form expression for these transfers when the decision rule satisfies order reducibility (See Section 7.4 in the Appendix). Assuming order reducibility, the minimally supermodular transfers are constructed from two equations: (7.41) and (7.42) (p.39 in the Appendix). The critical

[11] Note that K_i and γ_i depend on the type sets for every i. If the utility functions and the transfers are well-behaved, then K_i and γ_i exist in the limit as $\varepsilon(\Theta) \to 0$.

observation is that these transfers, and hence the K_i's they produce, depend entirely on the primitives of the model. In other words, the minimally supermodular transfers endogenize the K_i's. This property is very useful, because (5.5) becomes a condition that only involves the primitives of the design problem. As such, we can check it before building the mechanism and determine whether unique supermodular implementation might be attainable based on the primitives. By doing so, we learn valuable information about the type of environments for which supermodular implementation may be most useful. We explain this below.

For any implementable social choice function f, let

$$K_i^*(\theta) = \max_{\{\theta_i'',\theta_i',\theta_{-i}'',\theta_{-i}'\}} \frac{V_i(\theta_i' \triangleright \theta_i'', \theta_{-i}''; \theta) - V_i(\theta_i' \triangleright \theta_i'', \theta_{-i}'; \theta) - H_i(\theta_i'', \theta_i', \theta_{-i}'', \theta_{-i}')}{d_i(\theta_i'', \theta_i') \sum_{j \neq i} d_j(\theta_j'', \theta_j')} \tag{5.8}$$

where $V_i(\theta_i' \triangleright \theta_i'', \cdot; \theta) = V_i(x_i(\theta_i'', \cdot); \theta) - V_i(x_i(\theta_i', \cdot); \theta)$ and $H_i(\theta_i'', \theta_i', \theta_{-i}'', \theta_{-i}')$ is the sum of elements

$$\min_{\theta \in \Theta} [V_i(\hat{\theta}_i' \triangleright \hat{\theta}_i'', \hat{\theta}_{-i}''; \theta) - V_i(\hat{\theta}_i' \triangleright \hat{\theta}_i'', \hat{\theta}_{-i}'; \theta)]$$

for all pairs of immediate successors $\hat{\theta}_i'' \geq_i^1 \hat{\theta}_i'$, such that $\theta_i'' \geq_i^1 \hat{\theta}_i'' \geq_i^1 \hat{\theta}_i' \geq_i^1 \theta_i'$, and all pairs of immediate successors $\hat{\theta}_{-i}'' \geq_{-i} \hat{\theta}_{-i}'$ for a chosen "path" between θ_{-i}' and θ_{-i}'', such that $\theta_{-i}'' \geq_{-i} \hat{\theta}_{-i}'' \geq_{-i} \hat{\theta}_{-i}' \geq_{-i} \theta_{-i}'$. Notice that $\geq_i^1$ is a total order, which means that there is only one path of immediate successors connecting θ_i' and θ_i''. In contrast, $\geq_{-i}$ is a partial order and there could be many paths of immediate successors that connect any two ordered elements θ_{-i}' and θ_{-i}''. However, order reducibility ensures that irrespective of the path being chosen, the value of H_i will be the same.

Under the minimally supermodular transfers, $K_i^*(\theta)$ is the smallest number that satisfies (5.1). That is, $\bar{K}_i^*(\theta_i) \equiv \max_{\theta_{-i}} K_i^*(\theta)$ bounds the strategic complementarities in the game induced by these transfers at θ. When the designer uses the minimally supermodular transfers, $\bar{K}_i^*$ is the value that appears in condition (5.5). In this case, notice that (5.5) only involves the primitives of the model. If this inequality holds, then supermodular implementation is particularly well-suited for the design problem at hand, because the minimal transfers supermodularly implement the social choice function and truthtelling is eventually unique.

Expression (5.8) has a nice interpretation. It measures how much the supermodularity of the valuations varies across true types. We know that the designer must induce a supermodular game for any realization of types.[12] In this context, a large $\bar{K}_i^*$ can be

[12]It is sufficient but not necessary that the ex-post game be supermodular for each realization in order for the ex ante Bayesian game to be supermodular. For example, if the prior is mostly concentrated on some subset Θ' of Θ, it may not be necessary to make the ex-post payoffs supermodular for types in $\Theta \backslash \Theta'$. Of course, the possibility of neglecting $\Theta \backslash \Theta'$ depends on how unlikely that set is compared

caused by valuation functions that exhibit large substitutes for some (true) types (say θ) and large complementarities for other (true) types (say θ'). Since the designer does not know the realization of the true types, she will need to add a lot of complementarities through the transfers to ensure that the game is supermodular at θ. But this may induce a game that is "too supermodular" at θ', since there are already enough complementarities at that type, thereby violating (5.5) and causing multiplicity.

6 Conclusion

This paper extends supermodular mechanism design to environments with interdependent valuations, informational and allocative externalities, and arbitrary finite type spaces. While realistic, these environments present a serious challenge to mechanism designers. It is typically impossible to employ dominant strategy and ex-post equilibrium. This makes Bayesian implementation particularly relevant. In this context, supermodular Bayesian mechanisms are attractive.

The main motivation behind our mechanism design approach is to facilitate convergence to a desired equilibrium. This includes two problems: the robustness to bounded rationality (especially learning) and the multiple equilibrium problem. Supermodular mechanisms have nice learning properties, and the interval between their extremal equilibria contains all the limit points of learning dynamics. To some extent, this interval "measures" the multiple equilibrium problem. Our methodology is to impose orders on type sets, and given these orders, to induce a supermodular mechanism and to minimize its interval prediction by weakening the complementarities. It is worth mentioning that agents need not be aware of the orders. While the analyst can exploit the monotonicity of agents' best-responses to derive convergence properties, agents need not know or be informed that their best-responses are monotonic.

The paper has focused on behavioral robustness and left other issues unanswered.

First, our mechanisms are parametric. The designer needs to know the prior beliefs to construct the mechanisms, which is demanding (Ledyard (23)). Moreover, mistakes with respect to the prior may lead to shifts in equilibrium behavior and deviations from efficiency. Along this line, the literature on robust mechanism design (Bergemann and Morris (3)) advocates the use of ex-post equilibrium. But this is not always possible in these environments.

Second, we have avoided the issue of budget balancing. Robustness to bounded rationality may well come at the price of a balanced budget, i.e. full efficiency. In both of the examples presented in Section 2 and Section 7.1, the designer could achieve

to how submodular the utility function may be for types in that set.

dominance-solvability, uniqueness, and allocation efficiency by using the minimally supermodular transfers, but these transfers were not balanced. Reconciling budget balancing and minimal supermodularity (or, in general, dominance solvability) would be optimal, but this is an open question. If both properties were exclusive in general, the designer would be faced with a difficult choice: balancing the budget at the price of the implementation target (in case players do not learn to play the desired equilibrium), or guaranteeing the implementation target is reached at the price of a balanced budget.

7 Appendix

7.1 Another Motivating Example

Consider the motivating example of Section 2. The designer may choose the following transfers to implement the efficient decision rule:

$t_1(\hat{\theta})$	(1, 2)	(2, 1)
(1, 2)	2	-.10
(2, 1)	0	-.25

$t_2(\hat{\theta})$	(1, 2)	(2, 1)
(1, 2)	1.5	.25
(2, 1)	-1	.75

Given these transfers, the resulting payoff matrix for the ex ante Bayesian game is

EU	truthtelling	constant (1,2)	constant (2,1)	always lie
truthtelling	2.41*; .63*	2.38; .05	2.33*;0	2.29; -.13
constant (1,2)	2.33; 1.13	4.25*;1.25*	1.53; .25	3.45*; .38
constant (2,1)	2.38;-.63	1.63; -1	2; .5*	1.25; 0.13
always lie	2.29; -.13	3.5; -.25	1.2; .75*	2.41; .63

Both truthtelling and a constant announcement of (1,2) by both players are ex ante Bayesian equilibria. If we instead use the supermodular transfers that add minimal complementarities

$t_1^{SM}(\hat{\theta})$	(1, 2)	(2, 1)
(1, 2)	.95	.95
(2, 1)	-1.625	1.375

$t_2^{SM}(\hat{\theta})$	(1, 2)	(2, 1)
(1, 2)	.25	-.5
(2, 1)	.25	1.5

we obtain the ex ante payoff matrix:

EU	truthtelling	constant (1,2)	constant (2,1)	always lie
truthtelling	2.41*; .63*	1.04; .5	3.66*; 0	2.29; -.13
constant (1,2)	2.33; .13*	3.2*; 0	2.58; -.5	3.45*; -.63
constant (2,1)	2.38; .38	0; .25	3.63; 1.25*	1.25; 1.13
always lie	2.29; -.13	2.16; -.25	2.54; .75*	2.41; .63

Converting the original mechanism into a minimally supermodular mechanism has solved the multiple equilibrium problem. Truthtelling is the unique Bayesian equilibrium.

7.2 An Example of Linear System for Minimal Supermodular Implementation

Consider a setting with $n = 3$ agents, and types in $\Theta_i = \{1, 2\}$ for all i. Assume the conventional order $2 >_i^1 1$ for all i. For each player i, in order to minimally supermodular

implement the decision rule x, we are interested in finding a solution to the following system of linear equations:

$$\begin{pmatrix} 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \delta_i(1,(1,1)) \\ \delta_i(2,(1,1)) \\ \delta_i(1,(1,2)) \\ \delta_i(2,(1,2)) \\ \delta_i(1,(2,1)) \\ \delta_i(2,(2,1)) \\ \delta_i(1,(2,2)) \\ \delta_i(2,(2,2)) \end{pmatrix} = \begin{pmatrix} -\min_\theta X(\theta) \\ -\min_\theta Y(\theta) \\ -\min_\theta Z(\theta) \\ -\min_\theta W(\theta) \end{pmatrix}.$$

where

$$\begin{aligned} X(\theta) &= V_i(x_i(2,(1,2));\theta) - V_i(x_i(1,(1,2));\theta) - V_i(x_i(2,(1,1));\theta) + V_i(x_i(1,(1,1));\theta) \\ Y(\theta) &= V_i(x_i(2,(2,1));\theta) - V_i(x_i(1,(2,1));\theta) - V_i(x_i(2,(1,1));\theta) + V_i(x_i(1,(1,1));\theta) \\ Z(\theta) &= V_i(x_i(2,(2,2));\theta) - V_i(x_i(1,(2,2));\theta) - V_i(x_i(2,(1,2));\theta) + V_i(x_i(1,(1,2));\theta) \\ W(\theta) &= V_i(x_i(2,(2,2));\theta) - V_i(x_i(1,(2,2));\theta) - V_i(x_i(2,(2,1));\theta) + V_i(x_i(1,(2,1));\theta). \end{aligned}$$

Consider agent $i \in N$, whose valuations are given by:

$V_i(\cdot;\theta)$	(1,1,1)	(1,1,2)	(1,2,1)	(2,1,1)	(1,2,2)	(2,1,2)	(2,2,1)	(2,2,2)
A	0	0	0	0	0	0	0	0
B	3	1	2	1	2	2	2	2

Let us assume the decision rule to be implemented is:

	(1,1,1)	(1,1,2)	(1,2,1)	(2,1,1)	(1,2,2)	(2,1,2)	(2,2,1)	(2,2,2)
$\tilde{x}_i(\theta)$	B	A	A	B	B	B	B	B

Then the rhs of the system becomes:

$$\begin{pmatrix} -\min_\theta X \\ -\min_\theta Y \\ -\min_\theta Z \\ -\min_\theta W \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 3 \\ 3 \end{pmatrix}$$

One possible solution for the system is $\delta_i = (0,1,0,0,1,1,0,3)^T$.

7.3 Proofs

Proof of Theorem 1 Take a consistent profile of orders $\{(\geq_i^1, \geq_i^2)\}_i$. For every $i \in N$, each element $\theta_i \in \Theta_i$ is assigned an index k that corresponds to its position in the set Θ_i under the total order $\geq_i^1$. Similarly, each element $\theta_{-i} \in \Theta_{-i}$ is assigned an index q according to the total order order $\geq_i^2$ on Θ_{-i}. Suppose that $f = (x, t)$ is implementable. Letting

$$\delta_i(\hat{\theta}_i^k, \hat{\theta}_{-i}^q) \equiv -\sum_{l=1}^{k-1}\sum_{z=1}^{q-1} \min_{\theta\in\Theta}[V_i(x(\hat{\theta}_i^{l+1}, \hat{\theta}_{-i}^{z+1}); \theta) - V_i(x(\hat{\theta}_i^{l}, \hat{\theta}_{-i}^{z+1}); \theta) - V_i(x(\hat{\theta}_i^{l+1}, \hat{\theta}_{-i}^{z}); \theta) + V_i(x(\hat{\theta}_i^{l}, \hat{\theta}_{-i}^{z}); \theta)]. \tag{7.1}$$

for all $\hat{\theta}_i^k \in \Theta_i$ and $\hat{\theta}_{-i}^q \in \Theta_{-i}$, we define

$$t_i^*(\hat{\theta}_i^k, \hat{\theta}_{-i}^q) \equiv \delta_i(\hat{\theta}_i^k, \hat{\theta}_{-i}^q) - E_{\theta_{-i}}[\delta_i(\hat{\theta}_i^k, \theta_{-i})] + E_{\theta_{-i}}[t_i(\hat{\theta}_i^k, \theta_{-i})] \tag{7.2}$$

and show that (x, t^*) is minimally supermodular implementable.

Step 1. We show that t_i^* has smaller one-step supermodularity than any t_i such that (x, t) is supermodular implementable.

Let us define the one-step supermodularity of $V_i(x(\cdot); \theta)$ at any given announcement $(\hat{\theta}_i^k, \hat{\theta}_{-i}^q)$ as

$$g_i(k, q; \theta) \equiv V_i(x(\hat{\theta}_i^{k+1}, \hat{\theta}_{-i}^{q+1}); \theta) - V_i(x(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q+1}); \theta) - V_i(x(\hat{\theta}_i^{k+1}, \hat{\theta}_{-i}^{q}); \theta) + V_i(x(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q}); \theta). \tag{7.3}$$

For notational simplicity, we define

$$\begin{aligned} d_i(k, q) &\equiv \min_{\theta\in\Theta}[V_i(x(\hat{\theta}_i^{k+1}, \hat{\theta}_{-i}^{q+1}); \theta) - V_i(x(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q+1}); \theta) \\ &\quad -V_i(x(\hat{\theta}_i^{k+1}, \hat{\theta}_{-i}^{q}); \theta) + V_i(x(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q}); \theta)] \\ &= \min_{\theta\in\Theta} g_i(k, q; \theta). \end{aligned} \tag{7.4}$$

Since the one-step supermodularity of t_i^* is equivalent to the one-step supermodularity of δ_i we have

$$\begin{aligned} s_i(k, q) &\equiv \delta_i(\hat{\theta}_i^{k+1}, \hat{\theta}_{-i}^{q+1}) - \delta_i(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q+1}) - \delta_i(\hat{\theta}_i^{k+1}, \hat{\theta}_{-i}^{q}) + \delta_i(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q}) \\ &= -\sum_{l=1}^{k}\sum_{z=1}^{q} d_i(l, z) + \sum_{l=1}^{k-1}\sum_{z=1}^{q} d_i(l, z) + \sum_{l=1}^{k}\sum_{z=1}^{q-1} d_i(l, z) - \sum_{l=1}^{k-1}\sum_{z=1}^{q-1} d_i(l, z) \\ &= -d_i(k, q) \end{aligned} \tag{7.5}$$

as the one-step supermodularity of t_i^* (and δ_i).

Therefore, the one-step supermodularity of $(V_i + t_i^*)$ is given by

$$g_i(k, q; \theta) + s_i(k, q) \geq 0 \tag{7.6}$$

for all $\hat{\theta}_i^k, \hat{\theta}_{-i}^q$, θ, k, q, and i.

Denote the one-step supermodularity of transfer t_i as $sm_1(t_i; k, q)$, that is:

$$sm_1(t_i; k, q) = t_i(\hat{\theta}_i^{k+1}, \hat{\theta}_{-i}^{q+1}) - t_i(\hat{\theta}_i^k, \hat{\theta}_{-i}^{q+1}) - t_i(\hat{\theta}_i^{k+1}\hat{\theta}_{-i}^q) + t_i(\hat{\theta}_i^k, \hat{\theta}_{-i}^q).$$

For all transfers t such that (x, t) is supermodular implementable, it must hold that $g_i(k, q; \theta) + sm_1(t_i; k, q) \geq 0$ for all $\theta \in \Theta$, which is equivalent to:

$$\begin{aligned} sm_1(t_i; k, q) \geq -\min_{\theta \in \Theta} [V_i(x(\hat{\theta}_i^{k+1}, \hat{\theta}_{-i}^{q+1}); \theta) - V_i(x(\hat{\theta}_i^k, \hat{\theta}_{-i}^{q+1}); \theta) \\ - V_i(x(\hat{\theta}_i^{k+1}, \hat{\theta}_{-i}^q); \theta) + V_i(x(\hat{\theta}_i^k, \hat{\theta}_{-i}^q); \theta)] = s_i(k, q). \end{aligned} \tag{7.7}$$

The above shows that if (x, t) is supermodular implementable then the one-step supermodularity of transfers t is necessarily (weakly) greater than the one-step supermodularity of transfers t^*, which establishes Step 1.

Step 2. We show that the (multiple-step) supermodularity of any function of two variables is a sum of one-step supermodularities. Let us define the "(η, γ)-step supermodularity" of any function $t_i(\hat{\theta}_i^k, \hat{\theta}_{-i}^q)$ as

$$SM_{(\eta,\gamma)}(t_i; k, q) \;=\; t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q+\gamma}) \;-\; t_i(\hat{\theta}_i^k, \hat{\theta}_{-i}^{q+\gamma}) \;-\; t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^q) \;+\; t_i(\hat{\theta}_i^k, \hat{\theta}_{-i}^q). \tag{7.8}$$

Note that

$$\begin{aligned} t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q+\gamma}) = sm_1(t_i; k + \eta - 1, q + \gamma - 1) + t_i(\hat{\theta}_i^{k+\eta-1}, \hat{\theta}_{-i}^{q+\gamma}) \\ + t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q+\gamma-1}) - t_i(\hat{\theta}_i^{k+\eta-1}, \hat{\theta}_{-i}^{q+\gamma-1}), \end{aligned} \tag{7.9}$$

and so it follows from (7.8) that

$$\begin{aligned} SM_{(\eta,\gamma)}(t_i; k, q) = \big[sm_1(t_i; k + \eta - 1, q + \gamma - 1) + t_i(\hat{\theta}_i^{k+\eta-1}, \hat{\theta}_{-i}^{q+\gamma}) + t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q+\gamma-1}) \\ - t_i(\hat{\theta}_i^{k+\eta-1}, \hat{\theta}_{-i}^{q+\gamma-1})\big] - t_i(\hat{\theta}_i^k, \hat{\theta}_{-i}^{q+\gamma}) - t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^q) + t_i(\hat{\theta}_i^k, \hat{\theta}_{-i}^q)). \end{aligned} \tag{7.10}$$

Note that

$$t_i(\hat{\theta}_i^{k+\eta-1}, \hat{\theta}_{-i}^{q+\gamma}) = sm_1(t_i; k+\eta-2, q+\gamma-1) + t_i(\hat{\theta}_i^{k+\eta-2}, \hat{\theta}_{-i}^{q+\gamma}) + t_i(\hat{\theta}_i^{k+\eta-1}, \hat{\theta}_{-i}^{q+\gamma-1}) - t_i(\hat{\theta}_i^{k+\eta-2}, \hat{\theta}_{-i}^{q+\gamma-1}), \quad (7.11)$$

and therefore it follows from (7.10) that

$$\begin{aligned} SM_{(\eta,\gamma)}(t_i; k, q) &= sm_1(t_i; k+\eta-1, q+\gamma-1) + \Big[sm_1(t_i; k+\eta-2, g+\gamma-1) \\ &+ t_i(\hat{\theta}_i^{k+\eta-2}, \hat{\theta}_{-i}^{q+\gamma}) + t_i(\hat{\theta}_i^{k+\eta-1}, \hat{\theta}_{-i}^{q+\gamma-1}) - t_i(\hat{\theta}_i^{k+\eta-2}, \hat{\theta}_{-i}^{q+\gamma-1})\Big] \\ &+ t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q+\gamma-1}) - t_i(\hat{\theta}_i^{k+\eta-1}, \hat{\theta}_{-i}^{q+\gamma-1}) - t_i(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q+\gamma}) - t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q}) + t_i(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q}) \quad (7.12) \end{aligned}$$

which is equal to

$$\begin{aligned} &\sum_{n=1}^{2} sm_1(t_i; k+\eta-n, q+\gamma-1) + t_i(\hat{\theta}_i^{k+\eta-2}, \hat{\theta}_{-i}^{q+\gamma}) - t_i(\hat{\theta}_i^{k+\eta-2}, \hat{\theta}_{-i}^{q+\gamma-1}) \\ &\qquad + t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q+\gamma-1}) - t_i(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q+\gamma}) - t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q}) + t_i(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q}). \quad (7.13) \end{aligned}$$

Proceeding iteratively with this process of substitution and regrouping of terms for $n = (1, \ldots, \eta)$ we obtain

$$\begin{aligned} SM_{(\eta,\gamma)}(t_i; k, q) &= \sum_{n=1}^{\eta} sm_1(t_i; k+\eta-n, q+\gamma-1) + t_i(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q+\gamma}) - t_i(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q+\gamma-1}) \\ &\quad + t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q+\gamma-1}) - t_i(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q+\gamma}) - t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q}) + t_i(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q}) \\ &= \sum_{n=1}^{\eta} sm_1(t_i; k+\eta-n, q+\gamma-1) \\ &\quad + t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q+\gamma-1}) - t_i(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q+\gamma-1}) - t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q}) + t_i(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q}) \\ &= \sum_{n=1}^{\eta} sm_1(t_i; k+\eta-n, q+\gamma-1) + SM_{(\eta,\gamma-1)}(t_i; k, q). \quad (7.14) \end{aligned}$$

Iterating on Equation (7.14) for $m = 1, \ldots, \gamma - 1$ we obtain:

$$\begin{aligned} SM_{(\eta,\gamma)}(t_i; k, q) &= \sum_{n=1}^{\eta} sm_1(t_i; k + \eta - n, q + \gamma - 1) + SM_{(\eta,\gamma-1)}(t_i; k, q) \\ &= \sum_{n=1}^{\eta} sm_1(t_i; k + \eta - n, q + \gamma - 1) + \sum_{n=1}^{\eta} sm_1(t_i; k + \eta - n, q + \gamma - 2) + SM_{(\eta,\gamma-2)}(t_i; k, q) \\ &= \sum_{n=1}^{\eta} \sum_{m=1}^{\gamma-1} sm_1(t_i; k + \eta - n, q + \gamma - m) + SM_{(\eta,1)}(t_i; k, q). \end{aligned} \tag{7.15}$$

Now, using the fact that

$$\begin{aligned} SM_{(\eta,1)}(k, q) &= t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q+1}) - t_i(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q+1}) - t_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q}) + t_i(\hat{\theta}_i^{k}, \hat{\theta}_{-i}^{q}) \\ &= sm_1(t_i; k + \eta - 1, q) + SM_{(\eta-1,1)}(t_i; k, q) \\ &= \textstyle\sum_{n=1}^{\eta} sm_1(t_i; k + \eta - n, q) \end{aligned}$$

and plugging this into Equation (7.15), we obtain

$$\begin{aligned} SM_{(\eta,\gamma)}(t_i; k, q) &= \sum_{n=1}^{\eta} \sum_{m=1}^{\gamma-1} sm_1(t_i; k + \eta - n, q + \gamma - m) + \sum_{n=1}^{\eta} sm_1(t_i; k + \eta - n, q) \\ &= \sum_{n=1}^{\eta} \sum_{m=1}^{\gamma} sm_1(t_i; k + \eta - n, q + \gamma - m) \\ &= \sum_{l=k}^{k+\eta-1} \sum_{z=q}^{q+\gamma-1} sm_1(t_i; l, z). \end{aligned} \tag{7.16}$$

Thus, the multiple-step supermodularity of any function of two ordered variables is equal to the sum of one-step supermodularities, which establishes Step 2.

Step 3. Conclusion. Note that

$$E_{\theta_{-i}}[t_i^*(\hat{\theta}_i^k, \theta_{-i})] = E_{\theta_{-i}}[\delta_i(\hat{\theta}_i^k, \theta_{-i})] - E_{\theta_{-i}}[\delta_i(\hat{\theta}_i^k, \theta_{-i})] + E_{\theta_{-i}}[t_i(\hat{\theta}_i^k, \theta_{-i})] = E_{\theta_{-i}}[t_i(\hat{\theta}_i^k, \theta_{-i})] \tag{7.17}$$

and therefore transfers t_i and t_i^* have the same expected value given that all other agents report their types truthfully. That is, assuming truthful reporting, the expected utility of an agent is the same under t_i and t_i^*. Since (x, t) is truthfully implementable, the above implies that (x, t^*) is also truthfully implementable.

Using the result eshablished in Step 2, the (η, γ)-step supermodularity of $V_i(x(\cdot); \theta)$

at any given announcement $(\hat{\theta}_i^k, \hat{\theta}_{-i}^q)$ can now be written as:

$$\begin{aligned} G_i^{(\eta,\gamma)}(k,q;\theta) &= V_i(x(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q+\gamma});\theta) - V_i(x(\hat{\theta}_i^k, \hat{\theta}_{-i}^{q+\gamma});\theta) \\ &\quad -V_i(x(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^q);\theta) + V_i(x(\hat{\theta}_i^k, \hat{\theta}_{-i}^q);\theta) \\ &= \sum_{l=k}^{k+\eta-1} \sum_{z=q}^{q+\gamma-1} g_i(l,z;\theta). \end{aligned} \tag{7.18}$$

and the (η,γ)-step supermodularity of t_i^* is analogously given by

$$\begin{aligned} S_i^{(\eta,\gamma)}(k,q) &= \delta_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^{q+\gamma}) - \delta_i(\hat{\theta}_i^k, \hat{\theta}_{-i}^{q+\gamma}) - \delta_i(\hat{\theta}_i^{k+\eta}, \hat{\theta}_{-i}^q) + \delta_i(\hat{\theta}_i^k, \hat{\theta}_{-i}^q) \\ &= -\sum_{l=1}^{k+\eta-1} \sum_{z=1}^{q+\gamma-1} d_i(l,z) + \sum_{l=1}^{k-1} \sum_{z=1}^{q+\gamma-1} d_i(l,z) + \sum_{l=1}^{k+\eta-1} \sum_{z=1}^{q-1} d_i(l,z) - \sum_{l=1}^{k-1} \sum_{z=1}^{q-1} d_i(l,z) \\ &= -\sum_{l=k}^{k+\eta-1} \sum_{z=q}^{q+\gamma-1} d_i(l,z). \end{aligned} \tag{7.19}$$

It is straightforward to check that $G_i^{(\eta,\gamma)}(k,q;\theta)+S_i^{(\eta,\gamma)}(k,q) \geq 0$ for all $\hat{\theta}_i^k, \hat{\theta}_{-i}^q, \theta, k, q, \eta, \gamma$ and i and, therefore, t^* is supermodular implementable.

Moreover, Step 1 says that t^* has the smallest one-step supermodularity among all supermodular transfers t. Combined with Step 2, this establishes that t^* has the smallest (η,γ)-step supermodularity for any (η,γ) among all supermodular transfers t. Thus we conclude that (x,t^*) is minimally supermodular implementable under the chosen order profile $\{(\geq_i^1, \geq_i^2)\}_i$. Q.E.D

Proof of Corollary 1 In the proof of Theorem 1, we constructed transfers that minimally supermodular implemented the decision rule x under some chosen consistent profile of orders $\{(\geq_i^1, \geq_i^2)\}_i$. Each $(\geq_i^1, \geq_i^2)$ is a pair of complete orders on finite sets. Since there are finitely many agents, for each i there are finitely many complete orders, and consequently, finitely many consistent profiles. For each such profile, we can compute the distance between the largest and the smallest equilibrium in the ex ante induced game under the minimal transfers, using a metric d. Among all consistent profiles of orders we can thus choose the one associated with the smallest interval prediction as measured by d: denote this profile of orders by $\{(\geq_i^{*1}, \geq_i^{*2})\}_i$ and the corresponding minimal transfers by t^{**}. Therefore, t^{**} give the smallest interval prediction under d among all minimally supermodular transfers on consistent profiles of orders. Q.E.D

Proof of Proposition 2

$\Leftarrow$ **(sufficiency)** The structure of the system implies that if there exists a solution δ_i for all i, then for all θ_i', θ_i'', where θ_i'' is an immediate successor of θ_i' in Θ_i, and for all θ_{-i}', θ_{-i}'', where θ_{-i}'' is an immediate successor of θ_{-i}' in Θ_{-i}, the supermodularity of $\delta_i(\theta)$ is equal to

$$-\min_{\theta\in\Theta}[V_i(x_i(\theta_i'',\theta_{-i}''),\theta) - V_i(x_i(\theta_i'',\theta_{-i}'),\theta) - V_i(x_i(\theta_i',\theta_{-i}''),\theta) + V_i(x_i(\theta_i',\theta_{-i}'),\theta)]. \tag{7.20}$$

We construct transfers $t_i^*(\theta)$ by transforming any set of truth-revealing transfers $t_i(\theta)$ according to (4.2). Note that transfers $t_i(\theta)$ and $t_i^*(\theta)$ have the same expected value given that all other agents report truthfully. Hence transfers $t_i^*(\theta)$ also achieve truthful implementation of the same decision rule.

Moreover, it is easy to see from (4.2) that the supermodularity of $t_i^*(\theta)$ is equivalent to that of $\delta_i(\theta)$ for any two pairs of announcements. Since the one-step supermodularity of $\delta_i(\theta)$ is equivalent to that of $t_i^*(\theta)$, and in turn equal to the above equation, we can conclude that transfers $t_i^*(\theta)$ have the smallest possible one-step supermodularity.

To argue that transfers $t_i^*(\theta)$ also have the smallest multiple-step supermodularity, we rely on the proof of Theorem 1 (Step 2). In particular, (7.16) states that the multiple-step supermodularity of any function of two ordered variables is equal to the sum of all one-step supermodularities in between. We develop our argument below.

Take any $\theta_i'' \geq_i^1 \theta_i'$ and $\theta_{-i}'' \geq_{-i} \theta_{-i}'$. Consider the sequence of immediate successors $(\theta_i^0, \theta_i^1, \ldots, \theta_i^M)$ such that $\theta_i^0 = \theta_i'$, $\theta_i^M = \theta_i''$, and θ_i^{m+1} is the immediate successor of θ_i^m for all $m = 1, \ldots, M-1$. Since Θ_i is totally ordered by $\geq_i^1$, this sequence is unique. On the other hand, the set of opponent types Θ_{-i} is only partially ordered by $\geq_{-i}$. Thus, there may be several different sequences of immediate successors that connect θ_{-i}' to θ_{-i}''. Take any sequence of immediate successors $(\theta_{-i}^0, \theta_{-i}^1, \ldots, \theta_{-i}^N)$ such that $\theta_{-i}^0 = \theta_{-i}'$, $\theta_{-i}^N = \theta_{-i}''$, and θ_{-i}^n is an immediate successor of θ_{-i}^{n-1} for all $n = 1, \ldots, N-1$. We can now apply (7.16) to these sequences:

$$\begin{aligned}
&\delta_i(\theta_i'',\theta_{-i}'') - \delta_i(\theta_i'',\theta_{-i}') - \delta_i(\theta_i',\theta_{-i}'') + \delta_i(\theta_i',\theta_{-i}') = \\
&\qquad \delta_i(\theta_i^M,\theta_{-i}^N) - \delta_i(\theta_i^M,\theta_{-i}^0) - \delta_i(\theta_i^0,\theta_{-i}^N) + \delta_i(\theta_i^0,\theta_{-i}^0) = \\
&\qquad \sum_{m=1}^{M}\sum_{n=1}^{N}[\delta_i(\theta_i^m,\theta_{-i}^n) - \delta_i(\theta_i^m,\theta_{-i}^{n-1}) - \delta_i(\theta_i^{m-1},\theta_{-i}^n) + \delta_i(\theta_i^{m-1},\theta_{-i}^{n-1})].
\end{aligned} \tag{7.21}$$

Combining (7.20) and (7.21), we obtain:

$$\delta_i(\theta_i'',\theta_{-i}'') - \delta_i(\theta_i'',\theta_{-i}') - \delta_i(\theta_i',\theta_{-i}'') + \delta_i(\theta_i',\theta_{-i}') = \sum_{m=1}^{M}\sum_{n=1}^{N} - \min_{\theta\in\Theta}[V_i(x_i(\theta_i^m,\theta_{-i}^n),\theta) - V_i(x_i(\theta_i^m,\theta_{-i}^{n-1}),\theta) - V_i(x_i(\theta_i^{m-1},\theta_{-i}^n),\theta) + V_i(x_i(\theta_i^{m-1},\theta_{-i}^{n-1}),\theta)]. \quad (7.22)$$

Irrespective of which sequence of immediate successors connecting θ'_{-i} to θ''_{-i} we choose, the multiple-step supermodularity of $\delta_i(\theta)$, and therefore of $t_i^*(\theta)$, can always be represented as a sum of minimal one-step supermodularities. Thus, there are no other transfers that can do better on any multiple step while ensuring that all one steps are not smaller than the corresponding expression (7.20). Hence, transfers $t_i^*(\theta)$ have the smallest possible multiple-step supermodularities. We conclude $t \succeq_{\text{ID}} t^*$ for all $t \in \mathcal{T}$.

$\Rightarrow$ **(necessity)** Suppose the system does not have a solution. Then any collection of numbers $\{\delta_i(\theta) : \theta \in \Theta, i \in N\}$, and in particular any transfers $\{t_i(\theta) : \theta \in \Theta, i \in N\}$, must fall into one or both of the following cases:

Case 1: There exist i and two pairs of immediate successors (θ_i', θ_i'') and $(\theta'_{-i}, \theta''_{-i})$ such that

$$t_i(\theta_i'',\theta_{-i}'') - t_i(\theta_i'',\theta_{-i}') - t_i(\theta_i',\theta_{-i}'') + t_i(\theta_i',\theta_{-i}') > - \min_{\theta\in\Theta}[V_i(x_i(\theta_i'',\theta_{-i}''),\theta) - V_i(x_i(\theta_i'',\theta_{-i}'),\theta) - V_i(x_i(\theta_i',\theta_{-i}''),\theta) + V_i(x_i(\theta_i',\theta_{-i}'),\theta)]. \quad (7.23)$$

In this case, we can choose different numbers $\{\tilde{\delta}_i(\theta)\}$ such that, for these particular reports, the equality is satisfied:

$$\tilde{\delta}_i(\theta_i'',\theta_{-i}'') - \tilde{\delta}_i(\theta_i'',\theta_{-i}') - \tilde{\delta}_i(\theta_i',\theta_{-i}'') + \tilde{\delta}_i(\theta_i',\theta_{-i}') = - \min_{\theta\in\Theta}[V_i(x_i(\theta_i'',\theta_{-i}''),\theta) - V_i(x_i(\theta_i'',\theta_{-i}'),\theta) - V_i(x_i(\theta_i',\theta_{-i}''),\theta) + V_i(x_i(\theta_i',\theta_{-i}'),\theta)]. \quad (7.24)$$

Using numbers $\{\tilde{\delta}_i(\theta)\}$, we build transfers $\{\tilde{t}_i(\theta)\}$ according to equation (4.2). Clearly, it does not hold that $\tilde{t} \succeq_{\text{ID}} t$, which violates the definition of minimally supermodular implementation.

Case 2: There exist i and two pairs of immediate successors (θ_i', θ_i'') and $(\theta'_{-i}, \theta''_{-i})$

such that

$$t_i(\theta_i'',\theta_{-i}'')-t_i(\theta_i'',\theta_{-i}')-t_i(\theta_i',\theta_{-i}'')+t_i(\theta_i',\theta_{-i}')<$$
$$-\min_{\theta\in\Theta}[V_i(x_i(\theta_i'',\theta_{-i}''),\theta)-V_i(x_i(\theta_i'',\theta_{-i}'),\theta)-V_i(x_i(\theta_i',\theta_{-i}''),\theta)+V_i(x_i(\theta_i',\theta_{-i}'),\theta)]. \tag{7.25}$$

In this case, the supermodularity contained in t_i is not sufficient to induce a supermodular game, which also violates the definition of minimally supermodular implementation. Q.E.D

Proof of Proposition 3 By way of contradiction, suppose that profile $\theta^*(\cdot)\geq\theta^T(\cdot)$ is an equilibrium so that player i's best response to $\theta_{-i}^*(\cdot)$ is $\theta_i^*(\cdot)$. Thus, for all i, θ_i, and $\hat{\theta}_i$ such that $\theta_i^*(\theta_i)>_i^1\hat{\theta}_i\geq_i^1\theta_i$, the following must hold

$$E_{\theta_{-i}}[\Delta u_i(\theta_{-i}^*(\theta_{-i});\theta)]\equiv E_{\theta_{-i}}[u_i(\theta_i^*(\theta_i),\theta_{-i}^*(\theta_{-i});\theta)]-E_{\theta_{-i}}[u_i(\hat{\theta}_i,\theta_{-i}^*(\theta_{-i});\theta)]\geq 0. \tag{7.26}$$

We will show that this condition is not satisfied if the inequality in the theorem holds, i.e. there must be a player for whom a deception closer to the truthful strategy is strictly better than $\theta_i^*(\cdot)$. For simplicity, define[13]

$$E_{\theta_{-i}}[\Delta V_i(\theta_{-i};\hat{\theta}_i,\theta_{-i})]\equiv E_{\theta_{-i}}[V_i(x(\theta_i^*(\theta_i),\theta_{-i});\hat{\theta}_i,\theta_{-i})]-E_{\theta_{-i}}[V_i(x(\hat{\theta}_i,\theta_{-i});\hat{\theta}_i,\theta_{-i})]. \tag{7.27}$$

It follows from (5.1) and the definition of $\bar{K}_i(\theta_i)$ that for each i and θ_i:

$$E_{\theta_{-i}}[\Delta u_i(\theta_{-i}^*(\theta_{-i});\theta)]\leq E_{\theta_{-i}}[\Delta u_i(\theta_{-i};\theta)]+d_i(\theta_i^*(\theta_i),\hat{\theta}_i)\bar{K}_i(\theta_i)\sum_{j\neq i}E_{\theta_j}[d_j(\theta_j^*(\theta_j),\theta_j)]. \tag{7.28}$$

Since the social choice function (x,t) is implementable, the transfers $\{t_i\}$ induce truthful revelation. Therefore, it must be that for all i and θ_i the incentive compatibility constraint is satisfied, that is:

$$E_{\theta_{-i}}[V_i(x(\hat{\theta}_i,\theta_{-i});\hat{\theta}_i,\theta_{-i})]-E_{\theta_{-i}}[V_i(x(\theta_i^*(\theta_i),\theta_{-i});\hat{\theta}_i,\theta_{-i})]\geq$$
$$E_{\theta_{-i}}[t_i(\theta_i^*(\theta_i),\theta_{-i})]-E_{\theta_{-i}}[t_i(\hat{\theta}_i,\theta_{-i})]. \tag{7.29}$$

[13]The notation we used in Equation (5.8) becomes cumbersome in this proof, and so we replace $V_i(\hat{\theta}_i\triangleright\theta_i^*(\theta_i),\theta_{-i};\hat{\theta}_i,\theta_{-i})$ with $\Delta V_i(\theta_{-i};\hat{\theta}_i,\theta_{-i})$.

Thus, we obtain

$$\begin{aligned} E_{\theta_{-i}}[\Delta u_i(\theta_{-i};\theta)] &= E_{\theta_{-i}}[V_i(x(\theta_i^*(\theta_i),\theta_{-i});\theta_i,\theta_{-i})] - E_{\theta_{-i}}[V_i(x(\hat{\theta}_i,\theta_{-i});\theta_i,\theta_{-i})] \\ &\quad + E_{\theta_{-i}}[t_i(\theta_i^*(\theta_i),\theta_{-i})] - E_{\theta_{-i}}[t_i(\hat{\theta}_i,\theta_{-i})] \\ &\leq E_{\theta_{-i}}[\Delta V_i(\theta_{-i};\theta_i,\theta_{-i})] - E_{\theta_{-i}}[\Delta V_i(\theta_{-i};\hat{\theta}_i,\theta_{-i})] \\ &\leq -E_{\theta_{-i}}[\gamma_i(\theta_{-i})]d_i(\theta_i^*(\theta_i),\hat{\theta}_i)d_i(\hat{\theta}_i,\theta_i). \end{aligned} \tag{7.30}$$

where the first inequality is derived after substituting in the LHS of (7.29) and the second inequality follows from (5.3). Combining (7.28) and (7.30), we arrive at

$$\frac{E_{\theta_{-i}}[\Delta u_i(\theta_{-i}^*(\theta_{-i});\theta)]}{d_i(\theta_i^*(\theta_i),\hat{\theta}_i)} \leq \bar{K}_i(\theta_i)\sum_{j\neq i} E_{\theta_j}[d_j(\theta_j^*(\theta_j),\theta_j] - E_{\theta_{-i}}[\gamma_i(\theta_{-i})]d_i(\hat{\theta}_i,\theta_i). \tag{7.31}$$

If there exist i, θ_i, and $\hat{\theta}_i \in [\theta_i, \theta_i^*(\theta_i))$ such that

$$\bar{K}_i(\theta_i)\sum_{j\neq i} E_{\theta_j}[d_j(\theta_j^*(\theta_j),\theta_j] - E_{\theta_{-i}}[\gamma_i(\theta_{-i})]d_i(\hat{\theta}_i,\theta_i) < 0 \tag{7.32}$$

then by (7.31) $E_{\theta_{-i}}[\Delta u_i(\theta_{-i}^*(\theta_{-i});\theta)] < 0$, which contradicts (7.26). Therefore, $\theta^*(\cdot)$ is not a Bayesian equilibrium.

The same reasoning applies when $\theta^*(\cdot) \leq \theta^T(\cdot)$: if the condition of the theorem holds, $\theta^*(\cdot)$ cannot be a Bayesian equilibrium. Q.E.D

Proof of Proposition 4 Take any profile $\theta^*(\cdot) \geq \theta^T(\cdot)$ outside the neighborhood of truthtelling. By way of contradiction, suppose that $\theta^*(\cdot)$ is an equilibrium. Then, for all i and θ_i, the following must hold:

$$\begin{aligned} &E_{\theta_{-i}}[\Delta u_i(\theta_{-i}^*(\theta_{-i});\theta)] = \\ &\qquad E_{\theta_{-i}}[u_i(\theta_i^*(\theta_i),\theta_{-i}^*(\theta_{-i});\theta)] - E_{\theta_{-i}}[u_i(\hat{\theta}_i(\theta_i),\theta_{-i}^*(\theta_{-i});\theta)] \geq 0 \end{aligned} \tag{7.33}$$

for all deceptions $\hat{\theta}_i(\cdot) \in [\theta_i^T(\cdot), \theta_i^*(\cdot)]$. This in turn implies that the rhs of (7.31) must be nonnegative for all i, θ_i and $\hat{\theta}_i(\cdot)$, and thus its expected value (over θ_i) must also be nonnegative, that is

$$E_{\theta_i}[\bar{K}_i(\theta_i)]\sum_{j\neq i} E_{\theta_j}[d_j(\theta_j^*(\theta_j),\theta_j] - E_{\theta_{-i}}[\gamma_i(\theta_{-i})]E_{\theta_i}[d_i(\hat{\theta}_i(\theta_i),\theta_i)] \geq 0. \tag{7.34}$$

We will show that there is an agent i and a strategy $\hat{\theta}_i(\cdot)$ for which this inequality is violated, which is a contradiction.

Pick any agent i such that $E_{\theta_i}[d_i(\theta_i^*(\theta_i),\theta_i)] \geq E_{\theta_j}[d_j(\theta_j^*(\theta_j),\theta_j)]$ for all j. Let us show that agent i has an incentive to deviate from $\theta_i^*(\cdot)$ if the conditions of the proposition hold. By the definition of a metric,

$$E_{\theta_i}[d_i(\theta_i^*(\theta_i),\theta_i)] \leq E_{\theta_i}[d_i(\theta_i^*(\theta_i),\hat{\theta}_i(\theta_i))] + E_{\theta_i}[d_i(\hat{\theta}_i(\theta_i),\theta_i)]$$

and

$$E_{\theta_i}[d_i(\hat{\theta}_i(\theta_i),\theta_i)] \leq E_{\theta_i}[d_i(\theta_i^*(\theta_i),\hat{\theta}_i(\theta_i))] + E_{\theta_i}[d_i(\theta_i^*(\theta_i),\theta_i)],$$

and hence

$$|E_{\theta_i}[d_i(\theta_i^*(\theta_i),\theta_i)] - E_{\theta_i}[d_i(\hat{\theta}_i(\theta_i),\theta_i)]| \leq E_{\theta_i}[d_i(\theta_i^*(\theta_i),\hat{\theta}_i(\theta_i))] \tag{7.35}$$

for any $\hat{\theta}_i(\cdot)$. Note also that

$$\frac{\sum_{j\neq i} E_{\theta_j}[d_j(\theta_j^*(\theta_j),\theta_j]}{E_{\theta_i}[d_i(\theta_i^*(\theta_i),\theta_i)]} \leq (n-1) < \frac{E_{\theta_{-i}}[\gamma_i(\theta_{-i})]}{E_{\theta_i}[\bar{K}_i(\theta_i)]}, \tag{7.36}$$

where the first inequality follows from our choice of i, and the second inequality follows from the condition of the proposition. Hence,

$$E_{\theta_i}[\bar{K}_i(\theta_i)] \sum_{j\neq i} E_{\theta_j}[d_j(\theta_j^*(\theta_j),\theta_j)] - E_{\theta_{-i}}[\gamma_i(\theta_{-i})] E_{\theta_i}[d_i(\theta_i^*(\theta_i),\theta_i)] \tag{7.37}$$

$$\leq E_{\theta_i}[\bar{K}_i(\theta_i)] \sum_{j\neq i} E_{\theta_j}[d_j(\theta_j^*(\theta_j),\theta_j)] - E_{\theta_{-i}}[\gamma_i(\theta_{-i})] \frac{\sum_{j\neq i} E_{\theta_j}[d_j(\theta_j^*(\theta_j),\theta_j)]}{n-1}$$

$$= \sum_{j\neq i} E_{\theta_j}[d_j(\theta_j^*(\theta_j),\theta_j)] \left[E_{\theta_i}[K_i(\theta_i)] - \frac{E_{\theta_{-i}}[\gamma_i(\theta_{-i})]}{n-1} \right] < 0. \tag{7.38}$$

It follows from (7.35) and the definition of $\varepsilon(\Theta)$ that we can choose a deviation $\hat{\theta}_i(\cdot)$, which is closer to truthtelling, i.e. $\theta_i^*(\cdot) >_i \hat{\theta}_i(\cdot)$, and which is close enough to $\theta_i^*(\cdot)$ so that

$$E_{\theta_i}[d_i(\theta_i^*(\theta_i),\theta_i)] - \varepsilon(\Theta) \leq E_{\theta_i}[d_i(\hat{\theta}_i(\theta_i),\theta_i)]. \tag{7.39}$$

By (7.37) and (7.39), we obtain

$$E_{\theta_i}[\bar{K}_i(\theta_i)] \sum_{j\neq i} E_{\theta_j}[d_j(\theta_j^*(\theta_j),\theta_j)] - E_{\theta_{-i}}[\gamma_i(\theta_{-i})] E_{\theta_i}[d_i(\hat{\theta}_i(\theta_i),\theta_i)]$$

$$\leq E_{\theta_i}[\bar{K}_i(\theta_i)] \sum_{j\neq i} E_{\theta_j}[d_j(\theta_j^*(\theta_j),\theta_j)] - E_{\theta_{-i}}[\gamma_i(\theta_{-i})] \left[E_{\theta_i}[d_i(\theta_i^*(\theta_i),\theta_i)] - \varepsilon(\Theta) \right]$$

which is negative if

$$\sum_{j\neq i} E_{\theta_j}[d_j(\theta_j^*(\theta_j),\theta_j)]\left[\frac{E_{\theta_{-i}}[\gamma_i(\theta_{-i})]}{n-1} - E_{\theta_i}[K_i(\theta_i)]\right] > E_{\theta_{-i}}[\gamma_i(\theta_{-i})]\varepsilon(\Theta),$$

and thus, if

$$\min_i \left\{\sum_{j\neq i} E_{\theta_j}[d_j(\theta_j,\theta_j^*(\theta_j))]\right\} > \varepsilon(\Theta)\frac{\max_i\{E_{\theta_{-i}}[\gamma_i(\theta_{-i})]\}}{\min_i\left\{\frac{E_{\theta_{-i}}[\gamma_i(\theta_{-i})]}{n-1} - E_{\theta_i}[K_i(\theta_i)]\right\}}. \tag{7.40}$$

Therefore, (7.34) is violated, which means that $\hat{\theta}_i(\cdot)$ is a profitable deviation from $\theta_i^*(\cdot)$. Thus, $\theta^*(\cdot)$ is not an equilibrium. An analogous argument applies to the case $\theta^*(\cdot) \leq \theta^T(\cdot)$. Q.E.D

7.4 Unconditional Minimal Implementation and Order Reducibility

We first define a richness condition on decision rule x.

DEFINITION **9** *A decision rule $x(\theta)$ is order reducible if for each i, there are sets $\{G_p^i\}_{p=1}^P$ such that (a) $\Theta_{-i} = \cup_{p=1}^P G_p^i$, (b) for each θ_i, $x_i(\theta) = x_i(\theta_i,\theta'_{-i})$ for all $\theta_{-i},\theta'_{-i} \in G_p^i$, and (c) if $\theta_{-i} \in G_p^i$, all immediate successors of θ_{-i} must be in $G_p^i \cup G_{p+1}^i$.*

Order reducibility ensures that, through the structure of the decision rule, opponents' type profiles can be put into groups to form a linear path between the images of x_i. This linear path preserves the product order on Θ_{-i} and does not impose any ordering of images between unordered types. To illustrate the definition, consider a setting with $n = 3$ agents and $\Theta_i = \{1,2\}$ for all i. Assume types are ordered according to the usual order, i.e. $2 >_i^1 1$ for all i. Suppose the decision rule is $x_i(\theta) = x(\theta) = h(\sum \theta_i)$ where h is some strictly increasing real-valued function (Mathevet (26) presents several examples where the efficient decision rule takes this form). This decision rule is order reducible: for each agent i, it yields partition $G_1^i = \{(1,1)\}$, $G_2^i = \{(1,2),(2,1)\}$ and $G_3^i = \{(2,2)\}$. Note that for $n = 2$, order reducibility is trivially satisfied by all decision rules. Indeed, for each $j \neq i$, let each type in Θ_j form its own group with an index that corresponds to the position of the type under $>_j^1$. Below we present an example where order reducibility is violated.

PROPOSITION **5** *Let $f = (x,t)$ be a social choice function such that x is order reducible. If f is implementable, then there exist t^o such that (x,t^o) is minimally supermodular implementable.*

This proposition establishes minimal supermodular implementability of a class of social choice functions. For any implementable social choice function, if the decision rule satisfies order reducibility, then there exist transfers t^o that guarantee truthful supermodular implementation as well as the smallest equilibrium set among all supermodular transfers. There are many ways in which a mechanism can be converted into a supermodular one. It is therefore useful to describe the best way to convert it (and when it exists) given the objective of minimized equilibrium set. In the proof of the proposition, we provide an explicit formula for transfers t^o.

Order reducibility may seem to be a restrictive condition. Unfortunately, relaxing it just a little in a simple setting already defies existence of minimal transfers, as the following example demonstrates. Consider a three-agent two-type example. Let $\Theta_i = \{1, 2\}$ and $2 >_i^1 1$ for all i. Choose a decision rule x such that for some i, the only possible grouping is $G_1^i = \{(1, 1)\}$, $G_2^i = \{(1, 2)\}$, $G_3^i = \{(2, 1), (2, 2)\}$ (actual group indexes do not matter). This decision rule is not order reducible since (2,1), despite being an immediate successor of (1,1), is in a group that does not immediately follow G_i^1. For most valuation functions, a solution to our system of linear equations does not exist in this case. Thus, the transfers t^o are not minimal but no other transfers are.

Proof of Proposition 5 Suppose $f = (x, t)$ is implementable and x is order reducible. For every $i \in N$, assign to each element $\theta_i \in \Theta_i$ an index k that corresponds to its position in the set Θ_i under the total order $\geq_i^1$. Since x is order reducible, each element $\theta_{-i} \in \Theta_{-i}$ can be assigned an index p according to the group G_p^i to which it belongs. Note that more than one element θ_{-i} can be assigned the same index p, because all the elements in group G_p share the same index p. Letting

$$\delta_i(\theta_i^k, \theta_{-i}^p) = -\sum_{l=1}^{k-1}\sum_{z=1}^{p-1} \min_{\theta\in\Theta}[V_i(x(\theta_i^{l+1}, \theta_{-i}^{z+1}), \theta) - V_i(x(\theta_i^l, \theta_{-i}^{z+1}), \theta) - V_i(x(\theta_i^{l+1}, \theta_{-i}^z), \theta) + V_i(x(\theta_i^l, \theta_{-i}^z), \theta)] \tag{7.41}$$

for all $\theta_i^k \in \Theta_i$ and $\theta_{-i}^p \in \Theta_{-i}$, we define

$$t_i^o(\theta_i^k, \theta_{-i}^p) = \delta_i(\theta_i^k, \theta_{-i}^p) - E_{\theta_{-i}}[\delta_i(\theta_i^k, \theta_{-i})] + E_{\theta_{-i}}[t_i(\theta_i^k, \theta_{-i})] \tag{7.42}$$

and show that (x, t^o) is minimally supermodular implementable.

Note that $E_{\theta_{-i}}[t_i^o(\theta_i^k, \theta_{-i})] = E_{\theta_{-i}}[t_i(\theta_i^k, \theta_{-i})]$ and thus (x, t^o) is truthfully implementable. Moreover, the supermodularity of $t_i^o(\theta_i^k, \theta_{-i}^p)$ is equal to the supermodularity of $\delta_i(\theta_i^k, \theta_{-i}^p)$. We proceed to show in separate steps of the proof that transfers t^o achieve minimal supermodularities across immediate successors on $(\Theta_i, \geq_i^1)$ and

$(\Theta_{-i}, \geq_{-i})$ (Step 1) and that the supermodularities of t_i^o across (multiple-step) successive types are sums of supermodularities between immediate (one-step) successors (Step 2).

Step 1. Consider any two pairs of immediate successors $\theta_i'' \geq_i^1 \theta_i'$ and $\theta''_{-i} \geq_{-i} \theta'_{-i}$. As they are immediate successors, we can instead write $\theta_i^{k+1} \geq_i^1 \theta_i^k$. The (one-step) supermodularity of t_i^o is

$$t_i^o(\theta_i^{k+1}, \theta''_{-i}) - t_i^o(\theta_i^k, \theta''_{-i}) - t_i^o(\theta_i^{k+1}, \theta'_{-i}) + t_i^o(\theta_i^k, \theta'_{-i}) = \\ \delta_i(\theta_i^{k+1}, \theta''_{-i}) - \delta_i(\theta_i^k, \theta''_{-i}) - \delta_i(\theta_i^{k+1}, \theta'_{-i}) + \delta_i(\theta_i^k, \theta'_{-i}). \quad (7.43)$$

Since x is order reducible and $\theta''_{-i} \geq_{-i} \theta'_{-i}$ are immediate successors, it must be that either $\theta'_{-i}, \theta''_{-i} \in G_p^i$ or $\theta'_{-i} \in G_p^i$ and $\theta''_{-i} \in G_{p+1}^i$.

Case 1. If $\theta'_{-i}, \theta''_{-i} \in G_p^i$, then by order reducibility, $x(\theta_i, \theta'_{-i}) = x(\theta_i, \theta''_{-i})$ for all θ_i and we obtain

$$V_i(x(\theta_i^{k+1}, \theta''_{-i}); \theta) - V_i(x(\theta_i^k, \theta''_{-i}); \theta) - V_i(x(\theta_i^{k+1}, \theta'_{-i}); \theta) + V_i(x(\theta_i^k, \theta'_{-i}); \theta) = 0. \quad (7.44)$$

Using equation (7.41) for δ_i we have that $\delta_i(\theta_i, \theta'_{-i}) = \delta_i(\theta_i, \theta''_{-i}) = \delta_i(\theta_i, \theta_{-i}^p)$ for all θ_i. The supermodularity of t_i^o hence becomes:

$$t_i^o(\theta_i^{k+1}, \theta''_{-i}) - t_i^o(\theta_i^k, \theta''_{-i}) - t_i^o(\theta_i^{k+1}, \theta'_{-i}) + t_i^o(\theta_i^k, \theta'_{-i}) = \\ \delta_i(\theta_i^{k+1}, \theta_{-i}^p) - \delta_i(\theta_i^k, \theta_{-i}^p) - \delta_i(\theta_i^{k+1}, \theta_{-i}^p) + \delta_i(\theta_i^k, \theta_{-i}^p) = 0. \quad (7.45)$$

Hence, for all t_i such that (x, t) is supermodular implementable it must hold that:

$$t_i(\theta_i^{k+1}, \theta''_{-i}) - t_i(\theta_i^k, \theta''_{-i}) - t_i(\theta_i^{k+1}, \theta'_{-i}) + t_i(\theta_i^k, \theta'_{-i}) \geq \\ - \min_\theta [V_i(x(\theta_i^{k+1}, \theta''_{-i}); \theta) - V_i(x(\theta_i^k, \theta''_{-i}); \theta) - V_i(x(\theta_i^{k+1}, \theta'_{-i}); \theta) + V_i(x(\theta_i^k, \theta'_{-i}); \theta)] \\ = 0 = t_i^o(\theta_i^{k+1}, \theta''_{-i}) - t_i^o(\theta_i^k, \theta''_{-i}) - t_i^o(\theta_i^{k+1}, \theta'_{-i}) + t_i^o(\theta_i^k, \theta'_{-i}). \quad (7.46)$$

Therefore, for all i and immediate successors $\theta'_{-i}, \theta''_{-i} \in G_p^i$, transfers t_i^o have the smallest one-step supermodularity.

Case 2. If $\theta'_{-i} \in G_p^i$ and $\theta''_{-i} \in G_{p+1}^i$, using equation (7.41) to obtain the supermodularity

of t_i^o we get

$$\begin{aligned}
&t_i^o(\theta_i^{k+1},\theta''_{-i}) - t_i^o(\theta_i^k,\theta''_{-i}) - t_i^o(\theta_i^{k+1},\theta'_{-i}) + t_i^o(\theta_i^k,\theta'_{-i}) = \\
&\qquad \delta_i(\theta_i^{k+1},\theta_{-i}^{p+1}) - \delta_i(\theta_i^k,\theta_{-i}^{p+1}) - \delta_i(\theta_i^{k+1},\theta_{-i}^p) + \delta_i(\theta_i^k,\theta_{-i}^p) = \\
&-\min_\theta[V_i(x(\theta_i^{k+1},\theta_{-i}^{p+1});\theta) - V_i(x(\theta_i^k,\theta_{-i}^{p+1});\theta) - V_i(x(\theta_i^{k+1},\theta_{-i}^p);\theta) + V_i(x(\theta_i^k,\theta_{-i}^p);\theta)] = \\
&-\min_\theta[V_i(x(\theta_i^{k+1},\theta''_{-i});\theta) - V_i(x(\theta_i^k,\theta''_{-i});\theta) - V_i(x(\theta_i^{k+1},\theta'_{-i});\theta) + V_i(x(\theta_i^k,\theta'_{-i});\theta)].
\end{aligned} \tag{7.47}$$

Hence, for all t_i such that (x,t) is supermodular implementable it must hold that:

$$\begin{aligned}
&t_i(\theta_i^{k+1},\theta''_{-i}) - t_i(\theta_i^k,\theta''_{-i}) - t_i(\theta_i^{k+1},\theta'_{-i}) + t_i(\theta_i^k,\theta'_{-i}) \geq \\
&-\min_\theta[V_i(x(\theta_i^{k+1},\theta''_{-i});\theta) - V_i(x(\theta_i^k,\theta''_{-i});\theta) - V_i(x(\theta_i^{k+1},\theta'_{-i});\theta) + V_i(x(\theta_i^k,\theta'_{-i});\theta)] \\
&\qquad = t_i^o(\theta_i^{k+1},\theta''_{-i}) - t_i^o(\theta_i^k,\theta''_{-i}) - t_i^o(\theta_i^{k+1},\theta'_{-i}) + t_i^o(\theta_i^k,\theta'_{-i}).
\end{aligned} \tag{7.48}$$

Therefore, for all i and immediate successors $\theta'_{-i} \in G_p^i$ and $\theta''_{-i} \in G_{p+1}^i$, transfers t_i^o have the smallest one-step supermodularity.

Cases 1 and 2 allow us to conclude that transfers t^o achieve minimal supermodularities across any pair of immediate successors on $(\Theta_i, \geq_i^1)$ and $(\Theta_{-i}, \geq_{-i})$, as long as x is order reducible.

Step 2. Consider the supermodularity between successive types θ_i^k, θ_i^{k+q} and $\theta'_{-i} \in G_p^i$, $\theta''_{-i} \in G_{p+m}^i$. For $q = 1$ and $m = 1$ (or $m = 0$) this would reduce to the case of supermodularities between immediate successors considered in Step 1. Using equation (7.41), we obtain

$$\begin{aligned}
&t_i^o(\theta_i^{k+q},\theta''_{-i}) - t_i^o(\theta_i^k,\theta''_{-i}) - t_i^o(\theta_i^{k+q},\theta'_{-i}) + t_i^o(\theta_i^k,\theta'_{-i}) = \\
&\qquad \delta_i(\theta_i^{k+q},\theta_{-i}^{p+m}) - \delta_i(\theta_i^k,\theta_{-i}^{p+m}) - \delta_i(\theta_i^{k+q},\theta_{-i}^p) + \delta_i(\theta_i^k,\theta_{-i}^p) = \\
&\qquad -\sum_{l=k}^{k+q-1}\sum_{z=p}^{p+m-1}\min_{\theta\in\Theta}[V_i(x(\theta_i^{l+1},\theta_{-i}^{z+1}),\theta) - V_i(x(\theta_i^l,\theta_{-i}^{z+1}),\theta) \\
&\qquad\qquad - V_i(x(\theta_i^{l+1},\theta_{-i}^z),\theta) + V_i(x(\theta_i^l,\theta_{-i}^z),\theta)].
\end{aligned} \tag{7.49}$$

Hence, the q,m-step supermodularity of transfers t_i^o is a sum of all the one-step supermodularities between the groups G_p and G_{p+m}. We next show that this sum between the groups is equivalent to a sum of minimal one-step supermodularities on $\Theta_i \times \Theta_{-i}$, all of which need to be minimized for minimal supermodular implementation to hold.

Take a sequence $\theta_i^k, \ldots, \theta_i^{k+q}$ of immediate successors under $\geq_i^1$, and a sequence $\theta_{-i}^1, \ldots, \theta_{-i}^{1+s}$ of immediate successors under $\geq_{-i}$ such that $\theta_{-i}^1 = \theta'_{-i}$ and $\theta_{-i}^{1+s} = \theta''_{-i}$.

Since $\theta'_{-i} \in G^i_p$, $\theta''_{-i} \in G^i_{p+m}$, and x is order reducible, it cannot be that θ''_{-i} is more that s groups away from θ'_{-i}, i.e. it must be that $s \geq m$.

Case 1. If $m = s$, then

$$t_i^o(\theta_i^{k+q}, \theta''_{-i}) - t_i^o(\theta_i^k, \theta''_{-i}) - t_i^o(\theta_i^{k+q}, \theta'_{-i}) + t_i^o(\theta_i^k, \theta'_{-i}) = \tag{7.50}$$

$$- \sum_{l=k}^{k+q-1} \sum_{z=p}^{p+m-1} \min_{\theta \in \Theta} [V_i(x(\theta_i^{l+1}, \theta_{-i}^{z+1}), \theta) - V_i(x(\theta_i^l, \theta_{-i}^{z+1}), \theta)$$

$$- V_i(x(\theta_i^{l+1}, \theta_{-i}^z), \theta) + V_i(x(\theta_i^l, \theta_{-i}^z), \theta)] = \tag{7.51}$$

$$- \sum_{l=k}^{k+q-1} \sum_{w=1}^{s-1} \min_{\theta \in \Theta} [V_i(x(\theta_i^{l+1}, \hat{\theta}_{-i}^{w+1}), \theta) - V_i(x(\theta_i^l, \theta_{-i}^{w+1}), \theta)$$

$$- V_i(x(\theta_i^{l+1}, \theta_{-i}^w), \theta) + V_i(x(\theta_i^l, \theta_{-i}^w), \theta)]. \tag{7.52}$$

Since the supermodularity of V_i is equal to

$$\sum_{l=k}^{k+q-1} \sum_{w=1}^{s-1} [V_i(x(\theta_i^{l+1}, \theta_{-i}^{w+1}), \theta) - V_i(x(\theta_i^l, \theta_{-i}^{w+1}), \theta)$$

$$- V_i(x(\theta_i^{l+1}, \theta_{-i}^w), \theta) + V_i(x(\theta_i^l, \theta_{-i}^w), \theta)] \tag{7.53}$$

and all of the summands involve one-step supermodularities, it holds that

$$V_i(x((\theta_i^{k+q}, \theta''_{-i}), \theta) - V_i(x(\theta_i^k, \theta''_{-i}), \theta) - V_i(x(\theta_i^{k+q}, \theta'_{-i}), \theta) + V_i(x(\theta_i^k, \theta'_{-i}), \theta)]$$

$$+ [t_i^o(\theta_i^{k+q}, \theta''_{-i}) - t_i^o(\theta_i^k, \theta''_{-i}) - t_i^o(\theta_i^{k+q}, \theta'_{-i}) + t_i^o(\theta_i^k, \theta'_{-i})] \geq 0 \tag{7.54}$$

and the multiple-step supermodularity of t_i^o is the smallest possible, so that all one-steps are minimally supermodular.

Case 2. If $s > m$, it means that $s - m$ immediate successors $\tilde{\theta}''_{-i}$ under $\geq_{-i}$ are in the same category as their immediate predecessors $\tilde{\theta}'_{-i}$ and are disregarded in the sum (7.51). However, note that for all of these successors, it holds that:

$$V_i(x((\theta_i^{k+1}, \tilde{\theta}''_{-i}), \theta) - V_i(x(\theta_i^k, \tilde{\theta}''_{-i}), \theta) - V_i(x(\theta_i^{k+1}, \tilde{\theta}'_{-i}), \theta) + V_i(x(\theta_i^k, \tilde{\theta}'_{-i}), \theta)] = 0 \tag{7.55}$$

and hence equality between (7.51) and (7.52) prevails. The rest of the argument for this case follows that for case 1.

Steps 1 and 2 prove that transfers t_i^o minimally supermodular implement the decision rule x under the chosen profile of total orders $\{\geq_i^1\}_i$.

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